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DIFFERENTIAL EQUATIONS OF A NONLINEAR MULTIPOLAR ELEMENT

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Abstract. A method for forming nonlinear differential equations of a multipololar element, which connect its independent pole currents and independent polar voltages, is proposed. The difficulty of the analysis is that some of the internal and external unknowns may be under the symbol of differentiation. The starting information for this formation is the common differential equations of the system of internal and external currents and voltages. The method is demonstrated on the case of formation of the corresponding differential equations of the system as such that is formed by bipolar elements. The analysis is significantly simplified in the presence of internal *D*-degenerations of the system or resistive circuits.

Key words: nonlinear differential equations: multipolar element: independent pole currents and polar voltages.

1. Introduction

Any multipolar (n-pole) element is characterized in relation to the outer part of the circuit by a system of equations that express the relationships between its independent voltages and currents due to the nature of the element itself. These equations do not depend on the nature of the outer circuit. They must be satisfied both in the idle state of the element (when it is not included in the circuit at all), and in the state of short circuit (when all its poles are shorted together). The number of such equations for the selected system of independent side voltages and pole currents is in principle unlimited.

The equations of the element and the outer part of the circuit must provide the definition of all $n–s$ independent voltages of the parties and $n–s$ independent pole currents, where s is the number of electrically separated parts or channels in the case of the passable element. Independent side voltages and pole currents of the element and the outer part of the circuit will be chosen so that they can be common to both the element and the outer part of the circuit. Therefore, it is necessary that the equations of the element and the outer part of the circle taken together constitute a system of $2\times(n–s)$ independent equations. This is possible only when the element and the outer part of the circuit are characterized by $n–s$ independent equations. This is at the same time the largest number of equations of the element.

If the number of independent element equations were greater than $n–s$, then, based on arbitrarily selected $n–s$ equations, the calculation of the circuit would obtain different expressions for the side voltages and pole currents of the element, and this would contradict the experimentally established fact of unambiguous state of the circuit.

A multipolar element can have an arbitrary complex and unpredictable structure, the most complex physical processes can occur in it, therefor, the establishment of independent equations sometimes requires extraordinary ability and skill of the researcher. It is not always possible to describe a physical process with sufficient accuracy; in many cases it is necessary to be limited to approximate methods or to use experimental dependences. Sometimes deep knowledge of the physical process from related fields (physiccs, mechanics, heat engineering, chemistry, etc.) is required to describe the element. It is almost impossible to describe all unforeseen situations. But then we will consider a fairly universal approach, based on the description of the element as a calculation scheme built of bipolar elements.

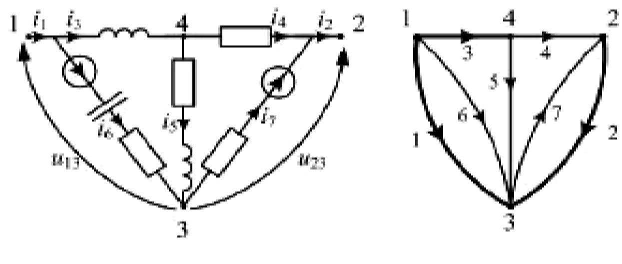

Fig. 1. Calculation scheme of the electric circuit of a tripole three-pole and its graph

Fig. 1 shows a calculation scheme of a three-pole element. It is characterized by a system of independent polar currents $I = (i_1, i_2)_t$ and a system of independent polar voltages $U_n = (u_{13}, u_{23})_t$. These currents and voltages belong to the external characteristics of the element, they represent the physical state of the element in the outer circuit. The remaining currents and voltages are internal and are not considered if possible.

The presence of nonlinear bipolar elements significantly complicates the task. Therefore, when constructing the equations of a multipolar nonlinear element, it is advisable to establish the quantitative relationship between independent currents and voltages through a system of algebraic-differential equations that contains both external and internal characteristics of the element.

Fig. 1 also shows a graph of the calculation scheme of the element and its selected tree. According to this graph, we form columns of currents and voltages of the edges λ_p $(\lambda = U, I) = (\lambda_1, \lambda_2, \lambda_3)_t$ and chords $\lambda_x =$ $= (\lambda = U, I) = (\lambda_4, \lambda_5, \lambda_6, \lambda_7)_t$.

Structural equations of the calculation scheme of the tripole will be [1, 2]

$$I_p + FI_x = 0; \quad -F_t U_p + U_x = 0, \tag{1}$$

where the topological matrix

$$F = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & \\ \hline -1 & & & -1 \\ \hline -1 & -1 & & \\ \hline \end{array}. \tag{2}$$

The equations of the branches are written (as obvious) in the form

$$\frac{di_3}{dt} = -\frac{u_3}{L_3}; \quad u_4 + r_4 i_4 = 0; \quad \frac{di_5}{dt} = -\frac{u_5 + r_5 i_5}{L_5};$$

$$u_6 + r_6 i_6 + u_{C6} - e_6 = 0; \quad \frac{du_{C6}}{dt} = -\frac{i_6}{C_6}; \tag{3}$$

$$u_7 + r_7 i_7 - e_7 = 0.$$

where L_3, L_5 are the differential inductances of the coils; C_5 is the dfferential capacitance of the capacitor; r_4, r_5, r_7 are static resistances of resistors.

The external voltages of the tripole three-pole are related to the voltages of the sides of the multipole by the obvious relations obtained on the basis of Kirchhof's law of voltages

$$u_{13} = -u_1; \; u_{23} = -u_2 . \tag{4}$$

The system of 15 algebraic-differential equations (2), (3), (4) contains 17 unknowns: $i_1,..., i_7$; u_1, ..., u_7; u_{C6}, u_{13}, u_{23}. It connects the columns of pole currents I and polar voltages $U_п$ and forms a complete system of equations of the output electric circuit. The remaining $(n–s)$ equations are obtained from the equations of the outer part of the circuit in which we include the element. If there is a need to investigate the element in an autonomous state, it is necessary to specify any two of the four external electromagnetic quantities of the element, for example, voltages u_{13}, u_{23} or currents i_1, i_2. Then the system of equations (1)–(4) will ensure the unambiguity of the solution, because it becomes completely definite.

Let's write it down in an expanded matrix form (5).

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & & & 1 & & & & & & & & & 1 & 1 & \\ \hline & 1 & & & & & & & & & & & -1 & & -1 \\ \hline & & L_3\frac{d}{dt} & & & & & 1 & & & & & & & \\ \hline & & & L_5\frac{d}{dt}+r_5 & & & & & & 1 & & & & & \\ \hline & & & & C_6\frac{d}{dt} & & & & & & & & & -1 & \\ \hline & & & & & -1 & & & & & 1 & & & & \\ \hline & & & & & & 1 & & & & & 1 & & & \\ \hline & & & & & -1 & & 1 & & 1 & & & & & \\ \hline & & & & & -1 & 1 & 1 & 1 & & & & & & \\ \hline & & & & & & & & 1 & & & & r_4 & & \\ \hline & & & & 1 & & & & & & 1 & & & r_6 & \\ \hline & & & & & & & & & & & 1 & & & r_7 \\ \hline & & 1 & -1 & & & & & & & & & -1 & & \\ \hline & & & & & -1 & & & & & & & & & \\ \hline & & & & & & -1 & & & & & & & & \\ \hline \end{array} \times \begin{array}{|c|} \hline i_1 \\ \hline i_2 \\ \hline i_3 \\ \hline i_5 \\ \hline u_{C6} \\ \hline u_1 \\ \hline u_2 \\ \hline u_3 \\ \hline u_4 \\ \hline u_5 \\ \hline u_6 \\ \hline u_7 \\ \hline i_4 \\ \hline i_6 \\ \hline i_7 \\ \hline \end{array} = \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline e_6 \\ \hline e_7 \\ \hline \\ \hline u_{13} \\ \hline u_{23} \\ \hline \end{array}. \tag{5}$$

Matrix equation (5) represents a system of output equations of the output electric circuit formed by bipolar elements. If, according to (3), (4), voltages u_1, u_2,..., u_7 and current i_6, are excluded from this system of equations, we obtain the corresponding system of hybrid equations of this tripole. We also write this system in the matrix form

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1 & & & 1 & C_6\dfrac{d}{dt} & 1 & \\ \hline & 1 & & & & -1 & -1 \\ \hline & & L_3\dfrac{d}{dt} & & & & \\ \hline & & L_3\dfrac{d}{dt} & L_5\dfrac{d}{dt}+r_5 & & & \\ \hline & & 1 & -1 & & -1 & \\ \hline & & & & & & r_7 \\ \hline -r_6 & & -r_6 & & 1 & & \\ \hline \end{array} \times \begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline i_3 \\ \hline i_5 \\ \hline u_{C6} \\ \hline i_4 \\ \hline i_7 \\ \hline \end{array} = \begin{array}{|c|}\hline \\ \hline \\ \hline u_{13}-u_{23} \\ \hline u_{13} \\ \hline \\ \hline e_7-u_{23} \\ \hline u_{13}+e_6 \\ \hline \end{array}. \qquad (6)$$

Having excluded currents: $i_4 = i_3 - i_5$; $i_7 = (e_7 - u_{23})/r_7$, $i_7 = (e_7 - u_{23})/r_7$ from the hybrid equations (6), we obtain

$$\begin{array}{|c|c|c|c|c|}\hline 1 & & 1 & & C_6\dfrac{d}{dt} \\ \hline & 1 & -1 & 1 & \\ \hline & & L_3\dfrac{d}{dt}+r_4 & -r_4 & \\ \hline & & L_3\dfrac{d}{dt} & L_5\dfrac{d}{dt}+r_5 & \\ \hline -r_6 & & -r_6 & & 1 \\ \hline \end{array} \times \begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline i_3 \\ \hline i_5 \\ \hline u_{C6} \\ \hline \end{array} = \qquad (7)$$

$$= \begin{array}{|c|}\hline \\ \hline (e_7-u_{23})/r_7 \\ \hline u_{13}-u_{23} \\ \hline u_{13} \\ \hline u_{13}+e_6 \\ \hline \end{array}.$$

Equations (7) connect the pole currents i_1, i_2 with the polar voltages u_{13}, u_{23}. They contain, in addition, three internal quantities i_3, i_5, u_{C6}, which cannot be excluded in the usual way, since they contain unknowns under the symbol of differentiation. But it is from this moment that the actual formation of the equations of a nonlinear multipole begins. That is why we offer an original approach, the essence of which is given below.

Differential equations that are subject to exclusion are translated into the status of voltage or current sources $u_C \to e_C, i_L \to j_L$. Since these values are obtained as a result of integrating separate equations, they should be transferred to the right-hand side of (7). Then their time derivatives $du_C/dt, di_L/dt$ are subject to exclusion. Therefore, we give equation (7) the corresponding form stated

$$\begin{array}{|c|c|c|c|c|}\hline -r_6 & & & & \\ \hline & 1 & & & \\ \hline & & L_3 & & \\ \hline & & L_3 & L_5 & \\ \hline 1 & & & & C_6 \\ \hline \end{array} \times \begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline di_3/dt \\ \hline di_5/dt \\ \hline du_{C6}/dt \\ \hline \end{array} - \qquad (8)$$

$$- \begin{array}{|c|}\hline u_{13}+e_6-e_{C6}+r_6 j_3 \\ \hline (e_7-u_{23})/r_7 + j_3 - j_5 \\ \hline u_{13}-u_{23}-r_4(j_3-j_5) \\ \hline u_{13}-r_5 j_5 \\ \hline j_3 \\ \hline \end{array}.$$

To generalize, we write system (8) in the matrix-block form

$$\begin{array}{|c|c|}\hline B_1 & B_2 \\ \hline B_3 & B_4 \\ \hline \end{array} \cdot \begin{array}{|c|}\hline x_1 \\ \hline x_2 \\ \hline \end{array} = \begin{array}{|c|}\hline y_1 \\ \hline y_2 \\ \hline \end{array}, \qquad (9)$$

where x_1, x_2 are respectively, the subcolumns of the pole currents of the element and derivatives to be excluded; y_1, y_2 are the subcolumns of the corresponding right parts.

Based on (9) you can write

$$\begin{gathered}(B_1 - B_2 B_4^{-1} B_3)x_1 = y_1 - B_2 B_4^{-1} y_2; \\ x_2 = B_4^{-1}\left(y_2 - B_3 x_1\right).\end{gathered} \qquad (10)$$

According to (8)–(10), we obtain the final equations of the multipole. After arranging the unknowns we will have

$$U = AI + E, \qquad (11)$$

where

$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \; I = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}; \; A = \begin{bmatrix} r_6 & \\ & -r_7 \end{bmatrix}; \qquad (12)$$

$$E = \begin{bmatrix} e_{C6} - e_6 - r_6 j_3 \\ e_7 + r_7 (j_3 - j_5) \end{bmatrix}.$$

The differential equations of the excluded derivatives will be

$$\frac{d}{dt}\begin{bmatrix} u_{C6} \\ i_3 \\ i_5 \end{bmatrix} = \begin{bmatrix} (u_{13} - u_{23} - r_4 (i_3 - i_5)) / L_3 \\ (u_{23} - (r_4 - r_5) i_5 - r_5 i_5) / L_5 \\ -(i_1 + i_3) / C_6 \end{bmatrix}. \qquad (13)$$

In the general case, the matrix A in (11) also contains differentiation symbols, if reactive elements are present in the pole current branches.

Equations (13) in the case of multipole operation in the system are integrated as independent. The required pole voltages are obtained from the equations of the system, which absorb the matrix-coefficients (12).

Note that in the presence of *EJ*-degeneracy, matrix B_4 in (10) can be special, for example, at $r_6 = 0$, in our example! In this case, you need to turn to the methods of solving degenerations of ideal energy sources [3].

In the case of a resistive nonlinear circuit, as already mentioned, there is no such problem of excluding internal electromagnetic quantities. It is also simplified in *D*-degenerate circuits, where the derivatives of all unknowns are present in the original equations of the multipole. In this case, in (9), it is sufficient to make a simple replacement $x_i \to dx_i / dt$, $i = 1, 2$ and further analysis becomes obvious [4]. Based on this, the problem of forming nonlinear differential equations of multipolar elements with totally *D*-degenerate internal electromagnetic circuits [4], which are electric machines and transformers of electromachine and electric power systems, was solved [5].

2. Conclusions

1. A method for forming nonlinear differential equations of a multipolar element, which connect its independent pole currents and independent polar voltages, based on the differential equations of a nondegenerate system of its external and internal voltages and currents is propozed.

2. The analysis is significantly simplified in cases of total *D*-degenerations of the system or resistive circuits.

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ДИФЕРЕНЦІАЛЬНІ РІВНЯННЯ НЕЛІНІЙНОГО БАГАТОПОЛЮСНОГО ЕЛЕМЕНТА

Чабан Василь

Запропоновано метод формування нелінійних диференціальних рівнянь багатополюсного елемента, які пов'язують між собою його незалежні полюсні струми і незалежні полярні напруги. Складність аналізу полягає у тому, що частина як внутрішніх, так і зовнішніх невідомих можуть перебувати під символом диференціювання. Стартовою інформацією для цього формування є спільні диференціальні рівняння системи внутрішніх і зовнішніх струмів та напруг. Метод продемонстровано на випадку формування відповідних диференціальних рівнянь системи як такої, що утворена двополюсними елементами. Аналіз істотно спрощується за наявності внутрішніх *D*-вироджень системи або резистивних кіл.

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