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AVERAGING OF ENTIRE FUNCTIONS OF IMPROVED REGULAR GROWTH WITH ZEROS ON A FINITE SYSTEM OF RAYS

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Using a Fourier series method for entire functions, we find an asymptotics of averaging of entire functions of improved regular growth with zeros on a finite system of rays.

Key words: entire function of improved regular growth, Fourier coefficients, finite system of rays.

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1. Introduction and main result

In [1, 2] (see also [3]) the class of entire functions of improved regular growth was introduced and a criteria for this regularity in the sense of zero distribution are established, when the zeros are located on a finite system of rays. In connection with the study of entire functions of improved regular growth with zeros on arbitrary system of rays, in [4] was proved the following statement.

Theorem A. *Let f be an entire function of order $\rho \in (0, +\infty)$ with the indicator h and let for some $\rho_1 \in (0, \rho)$ there exists an exceptional set $U \subset \mathbb{C}$ such that*

$$\log|f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_1}),\ U \not\ni z = re^{i\varphi} \to \infty, \quad (1)$$

and U can be covered by a system of pairwise disjoint disks $U_k = \{z : |z - a_k| < \tau_k\}$, $k \in \mathbb{N}$, satisfying

$$\sum_{k\in\mathbb{N}} \tau_k < +\infty, \quad \sum_{k\in\mathbb{N}} \tau_k |\log \tau_k| < +\infty.$$

Then there exists $\rho_2 \in (0, \rho)$ such that

$$J_f^r(\varphi) := \int_1^r \frac{\log|f(te^{i\varphi})|}{t}\, dt = \frac{r^\rho}{\rho} h(\varphi) + o(r^{\rho_2}) \quad (2)$$

as $r \to +\infty$ uniformly with respect to $\varphi \in [0, 2\pi]$.

In the present paper, using the Fourier series method [5] for the logarithm of the modulus of an entire function we shall prove the analog of Theorem A for entire functions of improved regular growth with zeros on a finite system of rays.

We remind the reader that an entire function f is said to be of *improved regular growth* (see [1, 2]), if for some $\rho \in (0, +\infty)$, $\rho_1 \in (0, \rho)$, and some 2π-periodic ρ-trigonometrically convex function $h(\varphi) \not\equiv -\infty$ there exists an exceptional set $U \subset \mathbb{C}$ such that relation (1) holds and U can be covered by a system of disks with finite sum of radii.

Remark that if f is an entire function of improved regular growth, then it has [1] the order ρ and indicator function h. Let us formulate our main result.

Theorem 1. *If an entire function f of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \ldots, m\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$, is of improved regular growth, then for some $\rho_2 \in (0, \rho)$ the relation (2) holds uniformly with respect to $\varphi \in [0, 2\pi]$.*

2. Preliminaries

Let f be an entire function with $f(0) = 1$, $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of its zeros, p is the smallest integer for which the series $\sum_{n=1}^\infty |\lambda_n|^{-p-1}$ converges. By

$$c_k(r, \log|f|) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} \log|f(re^{i\varphi})|\, d\varphi, \quad k \in \mathbb{Z},$$

$$c_k(r, J_f^r) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} J_f^r(\varphi)\, d\varphi, \quad k \in \mathbb{Z}, \quad r > 0,$$

we denote the Fourier coefficients of the functions $\log|f(re^{i\varphi})|$ and $J_f^r(\varphi)$, respectively.

Lemma 1. *If an entire function f of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \ldots, m\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$, is of improved regular growth, then there exists $\rho_3 \in (0, \rho)$ such that the asymptotic relation*

$$c_k(r, \log|f|) = \alpha_k r^\rho + \frac{o(r^{\rho_3})}{k^2 + 1}, \quad r \to +\infty, \quad (3)$$

holds uniformly for $k \in \mathbb{Z}$, where

$$\alpha_k := \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} h(\varphi)\, d\varphi$$

$$= \frac{\rho}{\rho^2 - k^2} \sum_{j=1}^m \Delta_j e^{-ik\psi_j}, \quad \Delta_j \in [0, +\infty),$$

for a noninteger ρ, and

$$\alpha_k = \begin{cases} \dfrac{\rho}{\rho^2-k^2}\sum\limits_{j=1}^{m}\Delta_j e^{-ik\psi_j}, & |k|\neq\rho=p, \\ \dfrac{\tau_f e^{i\theta_f}}{2} - \dfrac{1}{4\rho}\sum\limits_{j=1}^{m}\Delta_j e^{-i\rho\psi_j}, & k=\rho=p, \\ 0, & |k|\neq\rho=p+1, \\ \dfrac{Q_\rho}{2}, & k=\rho=p+1, \end{cases}$$

if ρ is an integer.

□ *Proof.* Let ρ is noninteger. Then [1] an entire function f, $f(0)=1$, can be represented in the form

$$f(z)=e^{Q(z)}\prod_{j=1}^{m}L_j(z), \tag{4}$$

where $Q(z):=\sum_{k=1}^{\nu}Q_kz^k$ is a polynomial of degree $\nu<\rho$ and $L_j(z)$ is the Weierstrass canonical product of genus p, $p=[\rho]<\rho<p+1$, constructed in zeros of the function f which lie on the ray $\{z:\arg z=\psi_j\}$. Since f is an entire function of improved regular growth, then [2] for some $\rho_4\in(0,\rho)$ and every $j\in\{1,\dots,m\}$

$$n_j(t)=\Delta_j t^{\rho}+o(t^{\rho_4}), \quad t\to+\infty, \quad \Delta_j\in[0,+\infty), \tag{5}$$

where $n_j(t)$ is the number of zeros of the function f from the disk $\{z:|z|\le t\}$, which are concentrated on a ray $\{z:\arg z=\psi_j\}$. Moreover, the indicator h of an entire function f of improved regular growth of noninteger order ρ has the form ([1])

$$h(\varphi)=\sum_{j=1}^{m}h_j(\varphi),$$

where $h_j(\varphi)$ is a 2π-periodic function such that on $[\psi_j,\psi_j+2\pi)$

$$h_j(\varphi)=\frac{\pi\Delta_j}{\sin\pi\rho}\cos\rho(\varphi-\psi_j-\pi).$$

Therefore,

$$\alpha_k=\frac{1}{2\pi}\sum_{j=1}^{m}\frac{\pi\Delta_j}{\sin\pi\rho}\left(\int\limits_{\psi_j}^{\psi_j+2\pi}e^{-ik\varphi}\cos\rho(\varphi-\psi_j-\pi)\,d\varphi\right)$$

$$=\frac{\rho}{\rho^2-k^2}\sum_{j=1}^{m}\Delta_j e^{-ik\psi_j}, \quad k\in\mathbb{Z}.$$

Further, in view of (4), we have (see [3,6])

$$c_k(r,\log|f|)=\overline{c_{-k}(r,\log|f|)}, \quad k\le -1, \tag{6}$$

$$c_0(r,\log|f|)=\sum_{j=1}^{m}N_j(r), \quad N_j(r)=\int\limits_0^r\frac{n_j(t)}{t}\,dt, \tag{7}$$

$$c_k(r,\log|f|)=\frac12 Q_kr^k+$$
$$+\frac{1}{2k}\sum_{j=1}^{m}\left(\sum_{\substack{0<|\lambda_n|\le r,\\ \arg\lambda_n=\psi_j}}\left[\left(\frac{r}{|\lambda_n|}\right)^k-\left(\frac{|\lambda_n|}{r}\right)^k\right]e^{-ik\psi_j}\right),$$
$$1\le k\le p, \tag{8}$$

and

$$c_k(r,\log|f|)=-\frac{1}{2k}\sum_{j=1}^{m}\left(\sum_{\substack{|\lambda_n|>r,\\ \arg\lambda_n=\psi_j}}\left(\frac{r}{|\lambda_n|}\right)^k+\right.$$
$$\left.+\sum_{\substack{0<|\lambda_n|\le r,\\ \arg\lambda_n=\psi_j}}\left(\frac{|\lambda_n|}{r}\right)^k\right)e^{-ik\psi_j}, \quad k\ge p+1. \tag{9}$$

Using (5), from (8), integrating by parts, for $1\le k\le p$ we obtain

$$c_k(r,\log|f|)=\frac12 Q_kr^k+$$
$$+\frac{1}{2k}\sum_{j=1}^{m}\left[\int\limits_0^r\left(\left(\frac{r}{t}\right)^k-\left(\frac{t}{r}\right)^k\right)dn_j(t)\right]e^{-ik\psi_j}$$
$$=\frac12 Q_kr^k+$$
$$+\frac{1}{2k}\sum_{j=1}^{m}\left[kr^k\int\limits_0^r\frac{n_j(t)}{t^{k+1}}\,dt+\frac{k}{r^k}\int\limits_0^r t^{k-1}n_j(t)\,dt\right]e^{-ik\psi_j}$$
$$=\frac{\rho r^\rho}{\rho^2-k^2}\sum_{j=1}^{m}\Delta_j e^{-ik\psi_j}+\frac{o(r^{\rho_4})}{k^2+1}$$
$$=\alpha_kr^\rho+\frac{o(r^{\rho_4})}{k^2+1}, \quad r\to+\infty. \tag{10}$$

Similarly, using formulas (5) and (9), for $k\ge p+1$ we get

$$c_k(r,\log|f|)$$
$$=-\frac{1}{2k}\sum_{j=1}^{m}\left(\int\limits_r^{+\infty}\left(\frac{r}{t}\right)^k dn_j(t)+\int\limits_0^r\left(\frac{t}{r}\right)^k dn_j(t)\right)e^{-ik\psi_j}$$
$$=\frac{-1}{2k}\sum_{j=1}^{m}\left(kr^k\int\limits_r^{+\infty}\frac{n_j(t)}{t^{k+1}}\,dt-\frac{k}{r^k}\int\limits_0^r t^{k-1}n_j(t)\,dt\right)e^{-ik\psi_j}$$
$$=\frac{\rho r^\rho}{\rho^2-k^2}\sum_{j=1}^{m}\Delta_j e^{-ik\psi_j}+\frac{o(r^{\rho_4})}{k^2+1}$$
$$=\alpha_kr^\rho+\frac{o(r^{\rho_4})}{k^2+1}, \quad r\to+\infty. \tag{11}$$

From (10), (11), (6) and (7), it follows (3).

Let now $\rho\in\mathbb{N}$. Then [2] an entire function f is of form (4), where $Q(z)$ is a polynomial of degree $\nu\le\rho$, p is the smallest integer such that $\sum_1^\infty|\lambda_n|^{-p-1}<+\infty$, and $L_j(z)$ is a Weierstrass canonical product of genus p, $p=\rho$ or $p=\rho-1$, constructed by the zeros of f which lie on a ray $\{z:\arg z=\psi_j\}$. Since f is an entire function of improved regular growth, then [2] for some

$\rho_4 \in (0,\rho)$ and every $j \in \{1,\dots,m\}$ the relation (5) holds, and, in addition, for some $\delta_f \in \mathbb{C}$ and $\rho_5 \in (0,\rho)$

$$\sum_{0<|\lambda_n|\le r} \lambda_n^{-\rho} = \delta_f + o(r^{\rho_5-\rho}), \quad r \to +\infty. \tag{12}$$

Besides, the indicator h of an entire function f of improved regular growth of order $\rho \in \mathbb{N}$ is defined by the formula ([2])

$$h(\varphi) = \begin{cases} \tau_f \cos(\rho\varphi + \theta_f) + \sum_{j=1}^{m} h_j(\varphi), & p = \rho, \\ Q_\rho \cos\rho\varphi, & p = \rho - 1, \end{cases} \tag{13}$$

where $\tau_f = |\delta_f/\rho + Q_\rho|$, $\theta_f = \arg(\delta_f/\rho + Q_\rho)$ and $h_j(\varphi)$ is a 2π-periodic function such that on $[\psi_j, \psi_j + 2\pi)$

$$h_j(\varphi) = \Delta_j(\pi - \varphi + \psi_j)\sin\rho(\varphi - \psi_j) - \frac{\Delta_j}{\rho}\cos\rho(\varphi - \psi_j).$$

First, let $p = \rho$. Then, according to (13), we get

$$\alpha_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\varphi}\tau_f \cos(\rho\varphi + \theta_f)\, d\varphi +$$

$$+\frac{1}{2\pi}\sum_{j=1}^{m}\Delta_j \int_{\psi_j}^{\psi_j+2\pi} e^{-ik\varphi}(\pi - \varphi + \psi_j)\sin\rho(\varphi - \psi_j)\, d\varphi -$$

$$-\frac{1}{2\pi}\sum_{j=1}^{m}\frac{\Delta_j}{\rho}\int_{\psi_j}^{\psi_j+2\pi} e^{-ik\varphi}\cos\rho(\varphi - \psi_j)\, d\varphi$$

$$= \frac{\rho}{\rho^2 - k^2}\sum_{j=1}^{m}\Delta_j e^{-ik\psi_j}, \quad |k| \ne \rho,$$

and

$$\alpha_\rho = \frac{\tau_f e^{i\theta_f}}{2} - \frac{1}{4\rho}\sum_{j=1}^{m}\Delta_j e^{-i\rho\psi_j}.$$

Further, (see [2, 3, 6]) the formula (8) holds for $1 \le k < p$, and formula (9) is true for $k \ge p+1$, and, in particular, for $k = p = \rho$ we have

$$c_\rho(r, \log|f|) = \frac{1}{2}Q_\rho r^\rho + \frac{1}{2\rho}\sum_{0<|\lambda_n|\le r}\left(\frac{r}{\lambda_n}\right)^\rho - \frac{1}{2\rho}\sum_{j=1}^{m} I(j), \tag{14}$$

where

$$I(j) = \sum_{\substack{0<|\lambda_n|\le r,\\ \arg\lambda_n = \psi_j}}\left(\frac{|\lambda_n|}{r}\right)^\rho e^{-i\rho\psi_j}.$$

Thus, in the same way as in the case of noninteger ρ, for $1 \le k < p$ and $k \ge p+1$ the asymptotic relation (3) holds. Let's consider the case $k = p = \rho$. Taking into account (5), we obtain

$$I(j) = \frac{e^{-i\rho\psi_j}}{r^\rho}\int_0^r t^\rho\, dn_j(t) = \frac{\Delta_j}{2} r^\rho e^{-i\rho\psi_j} + o(r^{\rho_4}) \tag{15}$$

as $r \to +\infty$. Combining (14), (15) and (12), we get

$$c_\rho(r, \log|f|) = \frac{r^\rho}{2}(Q_\rho + \delta_f/\rho) + o(r^{\rho_5}) -$$

$$-\frac{r^\rho}{4\rho}\sum_{j=1}^{m}\Delta_j e^{-i\rho\psi_j} + o(r^{\rho_4})$$

$$= \left(\frac{\tau_f e^{i\theta_f}}{2} - \frac{1}{4\rho}\sum_{j=1}^{m}\Delta_j e^{-i\rho\psi_j}\right) r^\rho + o(r^{\rho_6})$$

$$= \alpha_\rho r^\rho + o(r^{\rho_6}), \quad r \to +\infty, \quad 0 < \rho_6 < \rho.$$

Hence, in the case $k = p = \rho$ we also obtain (3). For other values of k the required statement follows from relations (6) and (7).

Now consider the case $\rho = p+1$. Taking into account (13), we get

$$\alpha_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\varphi} Q_\rho \cos\rho\varphi\, d\varphi = 0, \quad |k| \ne \rho,$$

and

$$\alpha_\rho = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\rho\varphi} Q_\rho \cos\rho\varphi\, d\varphi = \frac{Q_\rho}{2}.$$

Besides, (see [3, 6]) the formula (8) is fulfilled for $1 \le k \le p$, and formula (9) is true for $k > p+1$ and, in particular, for $k = p+1 = \rho$ we have

$$c_\rho(r, \log|f|) = \frac{1}{2}Q_\rho r^\rho - \frac{1}{2\rho}\sum_{j=1}^{m} I(j) -$$

$$-\frac{1}{2\rho}\sum_{j=1}^{m}\left(\sum_{\substack{|\lambda_n|>r,\\ \arg\lambda_n = \psi_j}}\left(\frac{r}{|\lambda_n|}\right)^\rho e^{-i\rho\psi_j}\right),$$

where $I(j)$ is defined above. Since [2] in this case the relation (5) holds with $\Delta_j = 0$, then likewise as in the case $p = \rho$, for $k \in \mathbb{Z}$, $k \ne p+1$, the relation (3) is valid. Let now $k = p+1 = \rho$. In view of (15), $I(j) = o(r^{\rho_4})$ as $r \to +\infty$, and it is easy to show that

$$\sum_{\substack{|\lambda_n|>r,\\ \arg\lambda_n = \psi_j}}\left(\frac{r}{|\lambda_n|}\right)^\rho e^{-i\rho\psi_j} = r^\rho e^{-i\rho\psi_j}\int_r^{+\infty}\frac{dn_j(t)}{t^\rho} = o(r^{\rho_4})$$

as $r \to +\infty$. Therefore

$$c_\rho(r, \log|f|) = \frac{1}{2}Q_\rho r^\rho + o(r^{\rho_4}) = \alpha_\rho r^\rho + o(r^{\rho_4}), \quad r \to +\infty.$$

The proof of Lemma 1 is thus completed. ■

Corollary 1. *Under the conditions of Lemma 1, we have*

$$c_k(r, J_f^r) = \alpha_k\frac{r^\rho}{\rho} + \frac{o(r^{\rho_3})}{k^2+1}, \quad r \to +\infty,$$

for some $\rho_3 \in (0, \rho)$ uniformly in $k \in \mathbb{Z}$.

Indeed, ([5, p. 112])

$$c_k(r, J_f^r) = \int\limits_0^r \frac{c_k(t, \log|f|)}{t}\, dt, \quad k \in \mathbb{Z},$$

whence the required proposition follows.

3. Proof of Theorem 1

Indeed, a Fourier series of the function $J_f^r(\varphi) - \dfrac{r^\rho}{\rho} h(\varphi)$ is the series

$$\sum_{k\in\mathbb{Z}} \left(c_k(r, J_f^r) - \frac{r^\rho}{\rho}\alpha_k \right) e^{ik\varphi}.$$

Then, according to Corollary 1, we get

$$\sum_{k\in\mathbb{Z}} \left(c_k(r, J_f^r) - \frac{r^\rho}{\rho}\alpha_k \right) e^{ik\varphi} = o(r^{\rho_3}), \quad r \to +\infty,$$

for some $\rho_3 \in (0, \rho)$ uniformly with respect to $\varphi \in [0, 2\pi]$, and the Theorem 1 is proved.

Corollary 2. *Let the hypotheses of Theorem 1 be satisfied. Then for some $\rho_2 \in (0, \rho)$*

$$\int\limits_1^r J_f^t(\varphi) \frac{dt}{t} = \frac{r^\rho}{\rho^2} h(\varphi) + o(r^{\rho_2}), \quad r \to +\infty,$$

uniformly with respect to $\varphi \in [0, 2\pi]$.

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УСРЕДНЕНИЕ ЦЕЛЫХ ФУНКЦИЙ УЛУЧШЕННОГО РЕГУЛЯРНОГО РОСТА С НУЛЯМИ НА КОНЕЧНОЙ СИСТЕМЕ ЛУЧЕЙ

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С помощью метода рядов Фурье для целых функций найдена асимптотика усреднений целых функций улучшенного регулярного роста с нулями на конечной системе лучей.

Ключевые слова: целая функция улучшеного регулярного возростания, коэффициенты Фурье, конечная система лучей.

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УСЕРЕДНЕННЯ ЦІЛИХ ФУНКЦІЙ ПОКРАЩЕНОГО РЕГУЛЯРНОГО ЗРОСТАННЯ З НУЛЯМИ НА СКІНЧЕННІЙ СИСТЕМІ ПРОМЕНІВ

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За допомогою методу рядів Фур'є для цілих функцій, знайдено асимптотику усереднень цілих функцій покращеного регулярного зростання з нулями на скінченній системі променів.

Ключові слова: ціла функція покращеного регулярного зростання, коефіцієнти Фур'є, скінченна система променів.

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