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MATHEMATICAL METHODS FOR CAD: THE METHOD OF PROPORTIONAL DIVISION OF THE WHOLE INTO TWO UNEQUAL PARTS

In this paper an analysis of the laws of quadratic irrationality of the roots of the quadratic equation $x^2 = px + q$ with modulus coefficients is described $p = q$, which describes the proportional division of the whole into two unequal parts and the characteristic equation of the second order recurrence relation. It is shown that in the phase diagram p, q there exists a set of irrational values of the roots with properties similar to those of the classical "golden" numbers $\varphi_+ = \frac{1}{2}\left(-1+\sqrt{5}\right) = +0.618...$ and $\Phi_+ = \frac{1}{2}\left(1+\sqrt{5}\right) = +1.618...$

Key words: golden ratio (GR), proportional division, the quadratic irrational

МАТЕМАТИЧНІ МЕТОДИ САПР: МЕТОД ПРОПОРЦІЙНОГО ПОДІЛУ ЦІЛОГО НА ДВІ НЕРІВНІ ЧАСТИНИ

Подано аналіз законів квадратичної ірраціональності коренів квадратного рівняння $x^2 = px + q$ з коефіцієнтами $p = q$, що описують пропорційний розподіл цілого числа на дві нерівні частини та характеристичне рівняння рекурентного співвідношення другого порядку. Показано, що на фазовій діаграмі p, q існує безліч ірраціональних значень коренів з властивостями, подібними до властивостей класичних "золотих" чисел $\varphi_+ = \frac{1}{2}\left(-1+\sqrt{5}\right) = +0.618...$ і $\Phi_+ = \frac{1}{2}\left(1+\sqrt{5}\right) = +1.618...$.

Ключові слова: золота пропорція (ЗП), пропорційний розподіл, квадратична ірраціональність.

Introduction

It is known that the mathematical support of multi-functional design is the basis of design, including micro-and nano-systems design. It involves solving labor-intensive problems related to the definition of the principles of constructing design objects and the evaluation of their properties based on the study of the processes of their functioning. The essence of the computer-aided functional design is in solving functional design problems using multi-functional mathematical models, design objects. The basic mathematical models for CAD were thoroughly developed and described in the seminal monographs [1, 2].

The golden section method occupies a significant place in the mathematical support of computer-aided design problems. Its specificity of dividing the whole into two unequal parts is the basis of the method of finding the function value, including the function maximum and minimum, therefore it is used to create effective optimization algorithms, as one of the principal operations research algorithms. In contrast to the classical Newton's method, the golden section method is not related to the differentiation, but only implies the replacement of the constant of the golden section with the ratio of the Fibonacci numbers, therefore it is easily transformed into the commonly known method of Fibonacci. One of the most important steps to optimizing is the formulation of a target function, the value of which is calculated with the growth of the design parameter in the interval of uncertainty, which is mainly subject to minimization. In terms of the golden section method, only two of each three values of the target function in the interval of uncertainty are further used, as the third one does not provide additional information.

Pacioli di Borgo was the first who described the golden ratio [3], Zeising [4; 5] was the first who revealed how it is expressed in segments, numbers and principles of forming the Fibonacci sequences, and Fechner systematized the findings [6]. The establishment of the mathematical organization called The Fibonacci Association in 1963 had a significant impact on the development of the golden ratio and the theory of Fibonacci and Lucas numbers. The Association began publishing a quarterly mathematical journal *The Fibonacci Quarterly* in 1963. One of the founders of the Fibonacci Association and *The Fibonacci Quarterly* was the American mathematician V. Hoggatt. Subsequently, the English mathematician R. Knott created the sites of the Fibonacci Association http://www.mscs.dal.ca/Fibonacci/ and “Fibonacci Numbers and the Golden Section”, and later the sites http: //britton.disted.camosun.bc.ca/goldslide/jbgoldslide.htm, http: //www.fhfriedberg and de / users / boergens / marken / beispiele / goldenerschnitt.htm.

The interest in the golden section method revived after the publication of the monographs [7] and [8], and today this method finds a wide range of different applications. After a successful application in discovering the structure of fullerenes the golden section method was used for designing quasi-crystalline lattices with the axis of symmetry of the fifth order [9–10], optical modulators on the Fibonacci structures [11], atomic systems [12-13], fine-structure constant [14], the simulation of the features of electronic processes in nano-structured materials [15], the theory of electric circles [[16–18], etc. [19–20].

1. Basic properties of proportional division

Many processes and phenomena are accompanied by the division of the characteristic L of a system. These include the redistribution between potential and kinetic energies in accordance with the law of conservation of mechanical energy, the intensity of the light flux in the process at the boundary of the section, the charge flux when transferred to a closed circle through active supports or at the point of branching, etc. Given one point of division of the system state with the coordinate x, the mathematical model of the so-called “golden” ratio (GR) of the relation between the unequal sections can be formulated as follows:

$$\frac{L}{x}=\frac{x}{L-x} \tag{1.1}$$

or as dimensionless units $\Phi=\frac{L}{x}$ and $\varphi=\frac{x}{L}$,

$$\Phi=\frac{1}{\Phi-1} \quad \text{or} \quad \frac{1}{\varphi}=\frac{\varphi}{1-\varphi}. \tag{1.2}$$

Then, the equation (1.2) can be rewritten as two quadratic equations with solutions:

$$\begin{cases}\Phi^2-\Phi-1=0,\\ \varphi^2+\varphi-1=0,\end{cases} (a) \Rightarrow \begin{cases}\Phi_{\pm}=\frac{1}{2}\left(1\pm\sqrt{5}\right)=\begin{cases}\Phi_{+}=+1.618...,\\ \Phi_{-}=-0.618...,\end{cases}\\ \varphi_{\pm}=-\frac{1}{2}\left(1\pm\sqrt{5}\right)=\begin{cases}\varphi_{+}=+0.618...,\\ \varphi_{-}=-1.618...\end{cases}\end{cases} (b) \tag{1.3}$$

The characteristic feature of the equations (1.3 (a)) is that the coefficients are equal in modulus and equal to unity, and of their solutions (1.3 (b)) only positive ones are of interest:

$$\begin{cases} \Phi_{+} = +1.618... = \Phi, \\ \varphi_{+} = +0.618... = \varphi, \end{cases} \tag{1.4}$$

for which based on the theorem of Vieta`s the so-called “golden” combinations are valid:

$$\Phi - \varphi = 1, \Phi \cdot \varphi = 1 \quad \text{and} \quad \frac{1}{\varphi} - \frac{1}{\Phi} = 1. \tag{1.5}$$

As shown in the studies [7, 8], the numbers Φ and φ are irrational in the equations (1.5) with coefficients equal in modulus. Indeed, if Φ and φ are expressed in terms of the ratio of two integers $\Phi = \dfrac{f}{g}$, then (1.3 (a)) gives the contradiction:

$$\begin{cases} \left(\dfrac{f}{g}\right)^2 - \dfrac{f}{g} - 1 = 0, \\ \left(\dfrac{f}{g}\right)^2 + \dfrac{f}{g} - 1 = 0, \end{cases} \Rightarrow \begin{cases} f(f-g) \neq g^2, \\ f(f+g) \neq g^2, \end{cases} \quad if \quad f, g - integer. \tag{1.6}$$

On the other hand, the equations (1.3(a)) give:

$$\begin{cases} \Phi = \dfrac{1}{\Phi - 1} = 1 + \dfrac{1}{\Phi} \Rightarrow \Phi \cdot (\Phi - 1) = 1, \\ \varphi = \dfrac{1}{\varphi + 1} = 1 - \dfrac{1}{\varphi} \Rightarrow \varphi \cdot (\varphi + 1) = 1. \end{cases} \tag{1.7}$$

Substituting the values Φ and φ for their values given by (1.7) on the right of (1.7) we get the numbers Φ and φ in the form of two types of chain fractions:

$$\begin{cases} \Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + ...}}, \\ \varphi = 1 - \cfrac{1}{1 - \cfrac{1}{1 - ...}}. \end{cases} \quad \text{and} \quad \begin{cases} \Phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + ...}}}, \\ \varphi = \sqrt{1 - \sqrt{1 - \sqrt{1 - ...}}}. \end{cases} \tag{1.8}$$

For the golden numbers Φ and φ, the laws of exponentiation are valid:

$$\begin{cases} \Phi^2 - \Phi - 1 = 0, \\ \varphi^2 + \varphi - 1 = 0, \end{cases} \Rightarrow \begin{cases} \Phi^3 - \Phi^2 = \Phi, \\ \varphi^3 + \varphi^2 = \varphi, \end{cases} ... \begin{cases} \Phi^n - \Phi^{n-1} = \Phi^{n-2}, \\ \varphi^n + \varphi^{n-1} = \varphi^{n-2}, \end{cases} \Rightarrow \begin{cases} \Phi^n = \Phi^{n-1} + \Phi^{n-2}, \\ \varphi^n = -\varphi^{n-1} + \varphi^{n-2}. \end{cases} \tag{1.9}$$

The quadratic equations of GR (1.3 (a)) can be formally transformed to the commonly known Pythagorean theorem:

$$\frac{L}{x} = \frac{x}{L - x} \Rightarrow L^2 = x^2 + \left(\sqrt{Lx}\right)^2 \Rightarrow \begin{cases} \Phi^2 - \Phi - 1 = 0 \Rightarrow \Phi^2 = \left(\sqrt{\Phi}\right)^2 + 1^2 \Rightarrow \left(\Phi, \sqrt{\Phi}, 1\right), \\ \varphi^2 + \varphi - 1 = 0 \Rightarrow 1^2 = \varphi^2 + \left(\sqrt{\varphi}\right)^2 \Rightarrow \left(1, \varphi, \sqrt{\varphi}\right). \end{cases} \tag{1.10}$$

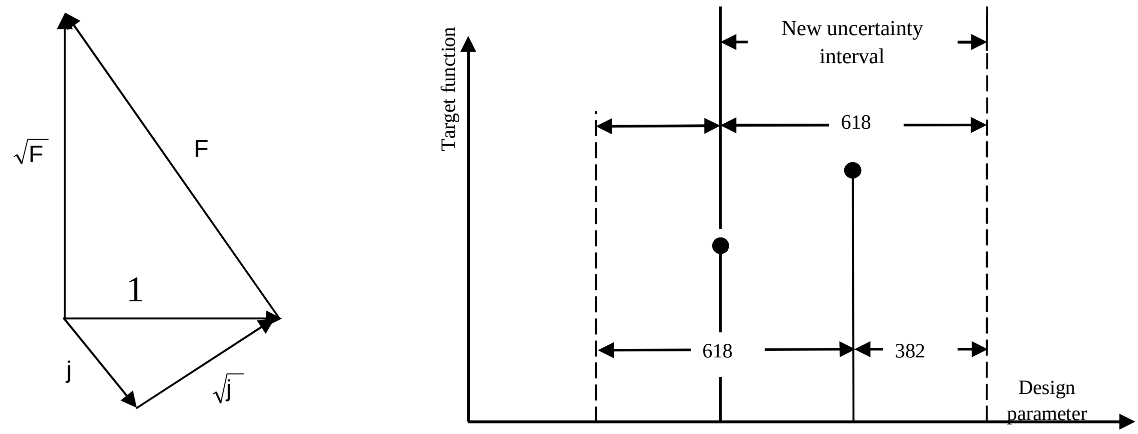
However, the equations coincide only formally. In the Pythagorean triangles with the hypotenuse Φ, the height $\sqrt{\Phi}$ and the base 1, and the hypotenuse 1, the height φ and the base $\sqrt{\varphi}$ (Fig. 1.1 (a)), the numbers Φ, $\sqrt{\Phi}$, 1 and 1, φ, $\sqrt{\varphi}$ express the coefficients of proportionality to the lengths of the sides of the triangle. The angle at the base is $\alpha = a\tan\left(\dfrac{\sqrt{1.618}}{1}\right) \cong 52^0$.

In the quadratic equations of GR (1.3 (a)) the values Φ and φ are the normalization coefficients for the lengths of the section parts of length L, to which the lengths of the segments are related as: $x = \dfrac{L}{\Phi} = \varphi L$. Therefore, the proportional equality of the golden section gives the quadratic equation:

$$\frac{L}{x} = \frac{x}{L-x} \Rightarrow x^2 + Lx - L^2 = 0 \Rightarrow \varphi^2 + \varphi - 1 = 0 \Rightarrow 1 = (\varphi)^2 + (\sqrt{\varphi})^2. \qquad (1.11)$$

Hence, the parts of the segment divided according to the GR are directly related by the quadratic equation $1 = (\varphi)^2 + (\sqrt{\varphi})^2$, therefore the Pythagorean golden triangle has parameters (1, φ, $\sqrt{\varphi}$). Unlike other quadratic equations, the equations (1.3 (a)) have coefficients equal in modulus and equal to unity.

$$x^2 + Lx - L^2 = 0 \Rightarrow x^2 + x - 1 = 0 \quad if \quad L^2 = L = 1 \Rightarrow L = 1. \qquad (1.12)$$



a *b*

Fig. 1.1. Geometry of the golden section of the uncertainty interval) (a) and the construction of the "golden" Pythagorean right triangles (b)*

The geometric interpretation of the Pythagorean theorem as a right triangle with parameters ($\Phi(1)$, $\sqrt{\Phi(1)}$, 1) is the basis of the well-known Kepler triangle, based on which the following spirals are composed the spiral of Theodorus and the spiral of Oster [21–23]. Long-term studies have established a wide range of regularities of quadratic irrationality and their connection with the proportional division of the whole [24–32]. The corresponding results were systematized in the monographs [7, 8, 18, 33–40].

2. A set of pairs of irrational numbers of the proportional division

The main property of the system of equations

$$\begin{cases} \Phi^2 - \Phi - 1 = 0, \\ \varphi^2 + \varphi - 1 = 0, \end{cases} \qquad (2.1)$$

is that their coefficients are equal in modulus and equal in this case to unity. The solutions to the equations (2.1) are irrational numbers $\Phi = 1.61803...$ and $\varphi = 0.61803...$. On the other hand, the equation (2.1) is a characteristic equation of the second-order recurrence relation

$$F(n) = F(n-1) + F(n-2) \qquad (2.2)$$

The implementation of GR for optimization of electric circle with consecutive and parallel resistors is viewed in the papers [10, 11].

In general, the second-order linear recurrent sequence (2.2) is given by the Fibonacci ratio with natural coefficients p and q [7, 8]

$$F(n) = pF(n-1) + qF(n-2), \tag{2.3}$$

and it has the characteristic equation

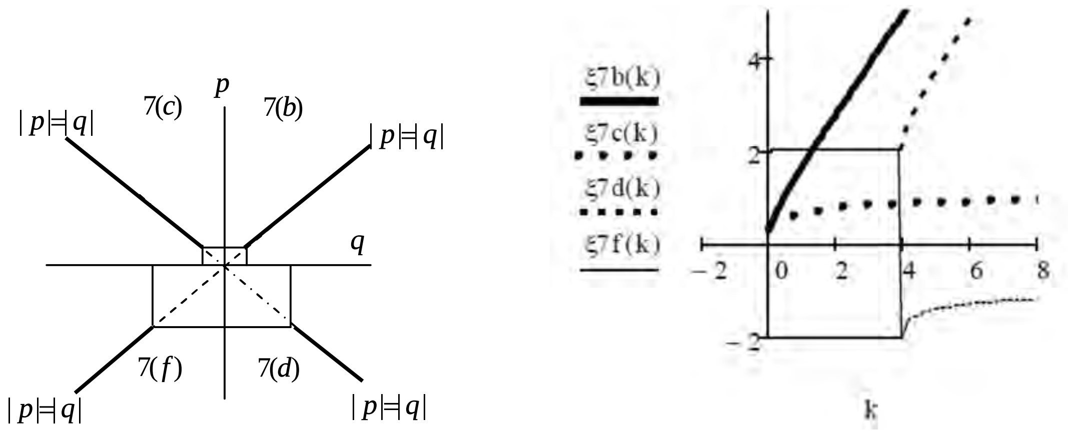
$$\xi^2 = p\xi + q,\ (q \neq 0), \tag{2.4}$$

with two solutions $x_{\pm}(p,q)$ (according to the classical GR $p=1,\quad q=1$).

Analyzing the solutions to the quadratic equation with coefficients equal in modulus, but unequal to $s = q = k$ in terms of the golden ratio, we study some general laws of equation (2.4):

$$\begin{cases} \xi^2 - p\xi - q = 0; \\ |p| = |q| = k > 0, \end{cases} (a) \quad \Rightarrow \quad \begin{cases} p>0,\ q>0 \ \Rightarrow \ \xi^2 - k\xi - k = 0,\ (b) - Case(b) \\ p>0,\ q<0 \ \Rightarrow \ \xi^2 - k\xi + k = 0,\ (c) - Case(c) \\ p<0,\ q>0 \ \Rightarrow \ \xi^2 + k\xi - k = 0,\ (d) - Case(d) \\ p<0,\ q<0 \ \Rightarrow \ \xi^2 + k\xi + k = 0,\ (f) - Case(f) \end{cases} \tag{2.5}$$

depending on the signs of the coefficients p and q. The ranges of values of the coefficients k , where the solutions to the quadratic equation (2.5 (a)) are valid for $k>0$ are represented by solid bold lines in Fig.1(a). In the upper half-plane $p>0$, the interval of positive values is $k > \frac{1}{2}$, as given $k < \frac{1}{2}$ a relative elongation $\frac{L}{x}$ is less than unity. In the lower half-plane $p<0$, the interval of allowed positive values is $k > 4$, as given $k < 4$ the roots of the quadratic equation are imaginary. Fig. 2.1 (b)) shows graphs of dispersion of solutions for different signs(p,q). In (2.5(f)), the solutions are negative, therefore they are not of interest.



a *b*

Fig. 2.1

For the characteristic parameter of proportional division $\Phi(k)$, based on (2.4), this equation is [42]:

$$\Phi(k)^2 - k\Phi(k) - k = 0 \tag{2.6}$$

at the same time, for $\Phi_{\pm}(k)$ the reciprocal value $\varphi_{\pm}(k)$ is given:

$$\varphi(k) = \frac{k}{\Phi(k)}. \tag{2.7}$$

A similar (2.6) quadratic equation for $\varphi(k)$ is represented as:

$$\Phi(k)^2 - k\Phi(k) - k = 0 \Rightarrow \frac{k^2}{\varphi(k)^2} - \frac{k^2}{\varphi(k)} - k = 0 \Rightarrow \varphi(k)^2 + k\varphi(k) - k = 0. \tag{2.8}$$

The solutions to the system of quadratic equation for $\Phi(k), \varphi(k)$

$$\begin{cases} \Phi(k)^2 - k\Phi(k) - k = 0, \\ \varphi(k)^2 + k\varphi(k) - k = 0, \end{cases} \tag{2.9}$$

are:

$$\Phi_{\pm}(k) = \frac{k}{2} \pm \sqrt{\frac{k^2}{4} + \sqrt{k^2}} = \begin{cases} \Phi_{+}(k) = \frac{k}{2} + \sqrt{\frac{k^2}{4} + k}, \\ \Phi_{-}(k) = \frac{k}{2} - \sqrt{\frac{k^2}{4} + k}, \end{cases} \tag{2.10}$$

and

$$\varphi_{\pm}(k) = -\frac{k}{2} \pm \sqrt{\frac{k^2}{4} + \sqrt{k^2}} = \begin{cases} \varphi_{+}(k) = -\frac{k}{2} + \sqrt{\frac{k^2}{4} + k}, \\ \varphi_{-}(k) = -\frac{k}{2} - \sqrt{\frac{k^2}{4} + k}. \end{cases} \tag{2.11}$$

For given $k = 1$, then $\Phi_{\pm}(1) = \frac{1}{2} \pm \sqrt{5}$ and $\varphi_{\pm}(1) = -\frac{1}{2} \pm \sqrt{5}$.

For real-life problems the following solutions are of interest

$$\Phi_{+}(k) = \frac{k}{2} + \sqrt{\frac{k^2}{4} + k} = \Phi(k) \quad \text{and} \quad \varphi_{+}(k) = -\frac{k}{2} + \sqrt{\frac{k^2}{4} + k} = \varphi(k). \tag{2.12}$$

Therefore, the properties characteristic of the golden section are substantiated for (2.12). According to the theorem of Vieta`s, the mathematical model of the GR for the roots (2.10) and (2.11) is represented as

$$\begin{matrix} \Phi_{+}(k) + \Phi_{-}(k) = k, \\ \Phi_{+}(k) \cdot \Phi_{-}(k) = -k, \end{matrix} (a) \quad and \quad \begin{matrix} \varphi_{+}(k) + \varphi_{-}(k) = -k, \\ \varphi_{+}(k) \cdot \varphi_{-}(k) = -k. \end{matrix} (b) \tag{2.13}$$

For positive solutions (2.13), the GR mathematical model is represented by the system of equations:

$$\begin{cases} \Phi(k) - \varphi(k) = k \quad and \quad \Phi(k) \cdot \varphi(k) = k, \\ \Phi(k) = k + \dfrac{k}{\Phi(k)} \quad and \quad \varphi(k) = \dfrac{k}{\varphi(k) + k}, \\ \Phi(k) = \sqrt{k + k\Phi(k)} \quad and \quad \varphi(k) = \sqrt{k - k\varphi(k)}. \end{cases} \tag{2.14}$$

The graphs of functions (2.14) are shown in Fig. 2.2 (a). Accordingly, a pair of numbers $\Phi = 1.618...$ and $\varphi = 0.618...$ with GR properties is not unique.

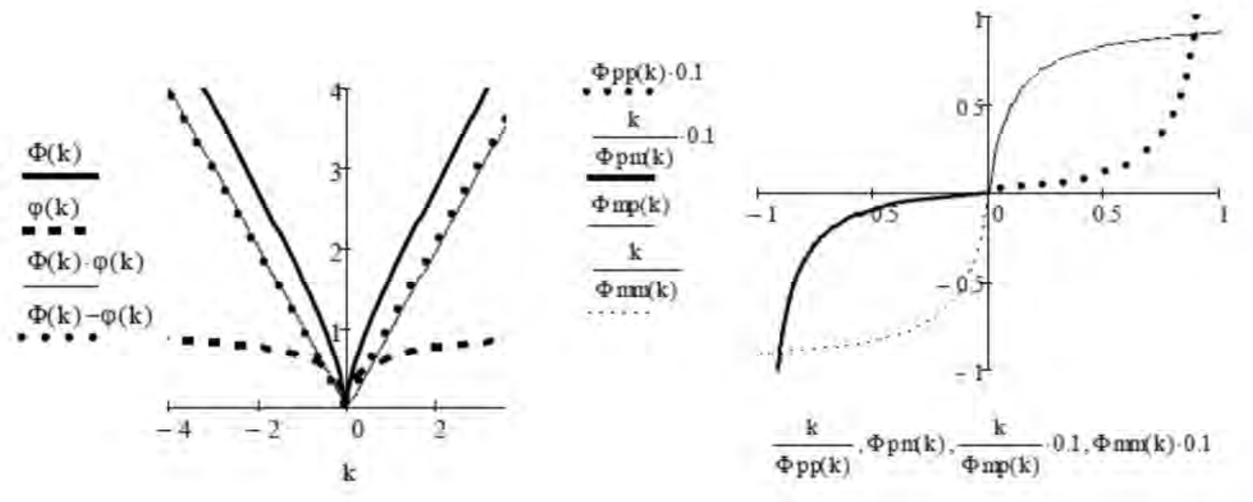
Let us prove the conclusion by specific calculations of the values $\Phi_{+}(k), \Phi_{-}(k), \varphi_{+}(k), \varphi_{-}(k)$. which are given for various coefficients k in Table 2.1. Here is an example of calculations of the lengths of the golden section of the segment with the length of $L = 1m$ for $k = 1$ and, for example $k = 7$. For $k = 1$ a golden constant is $\Phi(1) = 1.618...$, therefore the length of the longer segment is $\Phi(1) = \dfrac{L}{x_{k=1}} \Rightarrow x_{k=1} = \dfrac{1}{1.618...} = 0.618...$. The length of the smaller part is $L - x_{k=1} = 1 - 0.618... = 0.382...$. Given the golden section $k = 7$, $\Phi(7) = \dfrac{7 + \sqrt{7^2 + 4 \cdot 7}}{2} \cong 7.8875...$, the value of the coordinate of the point of division is $x_{k=7} = \dfrac{1}{7.8875...} \cong 0.1268...$, and respectively, the length of the second part is $L - x_{k=7} = 1 - 0.1268... = 0.8732...$. Thus, the basic properties of the GR (2.14) are fulfilled.

Table 2.1

Values $\Phi_+(k), \Phi_-(k), \varphi_+(k), \varphi_-(k)$ for different coefficients k

k	$\Phi_+(k)$	$\varphi_+(k)$	$\Phi_-(k)$	$\varphi_-(k)$
0.15	0.4695..	0.3195…	-0.3195…	-0.46958…
1	1.618..	0.618…	-0.618…	-1.618…
1.5	2.186…	0.686…	-0.686…	-2.186…
2	2.732…	0.732…	-0.732…	-2.732…
$\pi = 3.14$	3.937…	0.797…	-0.797…	-3.937…
111	111.991…	0.991…	-0.991…	-111.991…
1023	1024….	0.999…	-0.999…	-1024….

No restrictions were imposed on the values of the coefficients k. Therefore, as shown in Table 2.1, the roots (2.10) and (2.11) manifest the properties of the GR (2.14) with irrational values $k = 3.14$, $e = 2.71828$ (the basis of the natural logarithm) or $k = \frac{3}{2}$.



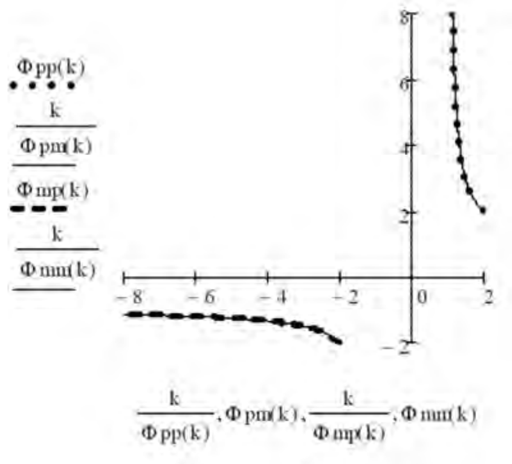
a *b*

Fig. 2.2. a – the graphs of functions (2.10) and (2.11);

$$b - \Phi sq(k) = \begin{cases} \Phi pp = 0.5(+k + \Lambda'), & \Phi mp = 0.5(-k + \Lambda'), \\ \Phi pm = 0.5(+k - \Lambda'), & \Phi mm = 0.5(-k - \Lambda'), \end{cases} \quad \Lambda' = \sqrt{\frac{k^2}{4} + k}$$

Fig. 2.1 (b) shows the graphs of solutions (2.10) and (2.11) in a parametric representation. Here $plus - plus\ (pp): p = +k, q = +k$, $\min us - \min us(mm): p = -k, q = -k$ is both coefficients $p > 0, q > 0$; $plus - \min us(pm): p = +k, q = -k$, $\min us - plus(mp): p = -k, q = +k$ is the signs of coefficients are different $p \gtrless 0, q \lessgtr 0$. The systematized ratios given in Table 2.2. reflect the impact of the change of signs of coefficients k on the form of dispersion curves. The first case has already been considered. In the second case, a dispersion curve continues the corresponding one for the first case and is represented by the mathematical model

$$\begin{cases} [\Phi_+(k)]^2 - [\Phi_-(k)]^2 = k, \\ \Phi_+(k) \cdot \Phi_-(k) = -k, \end{cases} \Leftrightarrow \quad [\Phi_-(k)]^2 - [\Phi_+(k)]^2 = \Phi_+(k) \cdot \Phi_-(k). \qquad (2.15)$$



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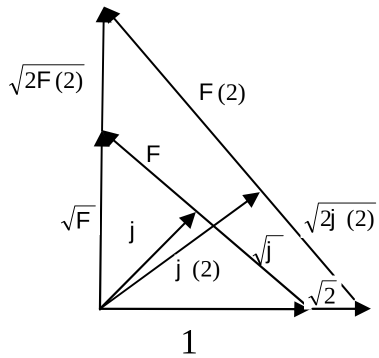

b

Fig.2.3. a – the graphs of functions $\Phi sq(k)=\begin{cases}\Phi pp = 0.5(+k+\Lambda), & \Phi mp = (-0.5k+\Lambda),\\ \Phi pm = 0.5(+k-\Lambda), & \Phi mm = (-0.5k-\Lambda),\end{cases}$ $\Lambda=\sqrt{\frac{k^2}{4}-k}$ *;*

b – the geometry of constructing the "golden" (2.15) Pythagorean right triangles for arbitrary positive values of the coefficient k *, except for* $k=0$ *(*$k \neq 0$*)*

Table 2.2

The impact of the change of signs of coefficients k on the form of dispersion curves.

$p \to +k$	$q \to +k$	$(2.13(a)):\ ^{+}_{-}k;$	k	$\Phi(k)$	$\varphi(k)$	$\Phi(k)-\varphi(k)$	$\Phi(k)\cdot\varphi(k)$
$p \to -k$	$q \to +k$	$(2.13(a)):\ ^{+}_{+}k;$	π	0.797...	3.937...	-3.14	$+3.14$
$p \to +k$	$q \to -k$	$(2.13(a)):\ ^{-}_{-}k;$	2π	5.032...	$-1.248...$	6.28	-6.28
$p \to -k$	$q \to -k$	$(2.13(a)):\ ^{-}_{+}k;$	2π	$-1.248...$	5.032...	-6.28	-6.28

In the third and fourth cases, the quadratic equation has real solutions only when $k \geq 4$. In the third case, the dispersion curves are localized in the first quadrant, and in the fourth case – in the third one (Fig. 2.2 (a)), which is mirror symmetric to the third one. We should also note that in the latter two cases, the dispersion curves originate at the point with coordinate $(\Phi=2, k=4)$. This restriction is caused by the condition $k \geq 4$. For these three cases of the values of the signs of the coefficients k Table 2.2 presents the calculations of numbers $\Phi(k)$ and $\varphi(k)$ for $k=3.14$ and, since in the latter two cases $k \geq 4$ is to be fulfilled, the value $k=6.28$ is used in calculations.

As follows from the conversion of ratios (2.15):

$$\begin{cases}\varphi(k)=\cfrac{k}{k+\cfrac{k}{k+\cfrac{k}{k+\dots}}}\\[2ex] \Phi(k)=k+\cfrac{k}{k+\cfrac{k}{k+\cfrac{k}{k+\dots}}}\end{cases} \quad and \quad \begin{cases}\varphi(k)=\sqrt{k-k\sqrt{k-k\sqrt{k-k\dots}}},\\ \Phi(k)=\sqrt{k+k\sqrt{k+k\sqrt{k+k\dots}}}\end{cases} \tag{2.16}$$

For $\varphi(k)$ is valid as a chain fraction for $\Phi(k)$ $\Phi(\pi)=\pi+\cfrac{\pi}{\pi+\cfrac{\pi}{+\cfrac{\pi}{3.937...}}} \cong 3.937...$ and

$\varphi(\pi)=\sqrt{\pi-\pi\cdot\sqrt{\pi-\pi\sqrt{\pi-\pi\cdot 3.791...}}}=3.791...$

Tables 2.3–2.6 show the systematic numerical values of solutions to quadratic equations $x^2 - px - q = 0$, depending on the signs of the factors p and q. In the tables 2.3 and 2.4 present the corresponding results for solutions $\Phi(p,q)$ and $\varphi(p,q)$ to the equation $\Phi(p,q)^2 - p\Phi(p,q) - q = 0$ with integer positive ($p>0$, $q>0$) values p and q .

Table 2.3

$\Phi(p,q)^2 - p\Phi(p,q) - q = 0$. The factors p, q are positive and integer numbers

p \ q	1	2	3	4	5	6	7
0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{7}$
1	$\frac{1+\sqrt{5}}{2} = 1.618$	$\frac{1+\sqrt{9}}{2} = 2$	$\frac{1+\sqrt{13}}{2} = 2.303$	$\frac{1+\sqrt{17}}{2} = 2.561$	$\frac{1+\sqrt{21}}{2} = 2.791$	$\frac{1+\sqrt{25}}{2} = 3$	$\frac{1+\sqrt{29}}{2} = 3.193$
2	$\frac{2+\sqrt{8}}{2} = 2.414$	$\frac{2+\sqrt{12}}{2} = 2.732$	$\frac{2+\sqrt{16}}{2} = 3$	$\frac{2+\sqrt{20}}{2} = 3.236$	$\frac{2+\sqrt{24}}{2} = 3.45$	$\frac{2+\sqrt{28}}{2} = 3.646$	$\frac{2+\sqrt{32}}{2} = 3.83$
3	$\frac{3+\sqrt{13}}{2} = 3.303$	$\frac{3+\sqrt{17}}{2} = 3.562$	$\frac{3+\sqrt{21}}{2} = 3.791$	$\frac{3+\sqrt{25}}{2} = 4$	$\frac{3+\sqrt{29}}{2} = 4.193$	$\frac{3+\sqrt{33}}{2} = 4.372$	$\frac{3+\sqrt{37}}{2} = 4.541$
4	$\frac{4+\sqrt{20}}{2} = 4.236$	$\frac{4+\sqrt{24}}{2} = 4.45$	$\frac{4+\sqrt{28}}{2} = 4.646$	$\frac{4+\sqrt{32}}{2} = 4.828$	$\frac{4+\sqrt{36}}{2} = 5$	$\frac{4+\sqrt{40}}{2} = 5.162$	$\frac{4+\sqrt{44}}{2} = 5.317$
5	$\frac{5+\sqrt{29}}{2} = 5.193$	$\frac{5+\sqrt{33}}{2} = 5.372$	$\frac{5+\sqrt{37}}{2} = 5.541$	$\frac{5+\sqrt{41}}{2} = 5.702$	$\frac{5+\sqrt{45}}{2} = 5.854$	$\frac{5+\sqrt{49}}{2} = 6$	$\frac{5+\sqrt{53}}{2} = 6.14$
6	$\frac{6+\sqrt{40}}{2} = 6.162$	$\frac{6+\sqrt{44}}{2} = 6.317$	$\frac{6+\sqrt{48}}{2} = 6.464$	$\frac{6+\sqrt{52}}{2} = 6.606$	$\frac{6+\sqrt{56}}{2} = 6.742$	$\frac{6+\sqrt{60}}{2} = 6.873$	$\frac{6+\sqrt{64}}{2} = 7$
7	$\frac{7+\sqrt{53}}{2} = 7.14$	$\frac{7+\sqrt{57}}{2} = 7.275$	$\frac{7+\sqrt{61}}{2} = 7.405$	$\frac{7+\sqrt{65}}{2} = 7.531$	$\frac{7+\sqrt{69}}{2} = 7.653$	$\frac{7+\sqrt{73}}{2} = 7.772$	$\frac{7+\sqrt{77}}{2} = 7.887$

Table 2.4

$\Phi(p,q)^2 - p\Phi(p,q) - q = 0$. The factors p, q

are positive and integer numbers

p \ q	1 $\Phi_{p,1} \times \varphi_{p,1} = 1$	2 $\Phi_{p,2} \times \varphi_{p,2} = 2$	3 $\Phi_{p,3} \times \varphi_{p,3} = 3$	4 $\Phi_{p,4} \times \varphi_{p,4} = 4$	5 $\Phi_{p,5} \times \varphi_{p,5} = 5$	6 $\Phi_{p,6} \times \varphi_{p,6} = 6$	7 $\Phi_{p,7} \times \varphi_{p,7} = 7$
1	2	3	4	5	6	7	8
0	$\Phi = 1$ $\varphi = 1$	$\Phi = 1.414$ $\varphi = 1.414$	$\Phi = 1.732$ $\varphi = 1.732$	$\Phi = 2$ $\varphi = 2$	$\Phi = 2.236$ $\varphi = 2.236$	$\Phi = 2.45$ $\varphi = 2.45$	$\Phi = 2.646$ $\varphi = 2.646$
1	$\Phi = 1.618$ $\varphi = 0.618$	$\Phi = 2$ $\varphi = 1$	$\Phi = 2.303$ $\varphi = 1.303$	$\Phi = 2.561$ $\varphi = 1.561$	$\Phi = 2.79$ $\varphi = 1.79$	$\Phi = 3$ $\varphi = 2$	$\Phi = 3.193$ $\varphi = 2.193$
2	$\Phi = 2.414$ $\varphi = 0.414$	$\Phi = 2.732$ $\varphi = 0.732$	$\Phi = 3$ $\varphi = 1$	$\Phi = 3.236$ $\varphi = 1.236$	$\Phi = 3.45$ $\varphi = 1.45$	$\Phi = 3.646$ $\varphi = 1.646$	$\Phi = 3.83$ $\varphi = 1.83$
3	$\Phi = 3.303$ $\varphi = 0.303$	$\Phi = 3.562$ $\varphi = 0.562$	$\Phi = 3.79$ $\varphi = 0.79$	$\Phi = 4$ $\varphi = 1$	$\Phi = 4.19$ $\varphi = 1.19$	$\Phi = 4.37$ $\varphi = 1.37$	$\Phi = 4.54$ $\varphi = 1.54$

Continuation tab. 2.4

1	2	3	4	5	6	7	8
4	$\Phi=4.236$ $\varphi=0.236$	$\Phi=4.45$ $\varphi=.45$	$\Phi=4.646$ $\varphi=.646$	$\Phi=4.828$ $\varphi=.828$	$\Phi=5$ $\varphi=1$	$\Phi=5.162$ $\varphi=1.162$	$\Phi=5.317$ $\varphi=1.317$
5	$\Phi=5.193$ $\varphi=0.193$	$\Phi=5.37$ $\varphi=0.37$	$\Phi=5.54$ $\varphi=0.54$	$\Phi=5.70$ $\varphi=0.70$	$\Phi=5.854$ $\varphi=0.854$	$\Phi=6$ $\varphi=1$	$\Phi=6.14$ $\varphi=1.14$
6	$\Phi=6.16$ $\varphi=0.16$	$\Phi=6.317$ $\varphi=0.317$	$\Phi=6.464$ $\varphi=0.464$	$\Phi=6.606$ $\varphi=0.606$	$\Phi=6.74$ $\varphi=0.74$	$\Phi=6.873$ $\varphi=0.873$	$\Phi=7$ $\varphi=1$
7	$\Phi=7.14$ $\varphi=0.14$	$\Phi=7.275$ $\varphi=0.275$	$\Phi=7.405$ $\varphi=0.405$	$\Phi=7.53$ $\varphi=0.53$	$\Phi=7.653$ $\varphi=0.653$	$\Phi=7.77$ $\varphi=0.77$	$\Phi=7.887$ $\varphi=0.887$

From the results presented in Tables 2.3 and 2.4, the following general patterns can be given

1. The positive roots of the equation $\Phi(p,q)^2 - p\Phi(p,q) - q = 0$ with the integer positive factors p, q are integer and irrational numbers.

2. For each value of the factor p_N (horizontal line number in the tables), the integer values $\Phi(p,q) = N$ of the roots are determined from the equation

$$\Phi(p,q) = N = \frac{1}{2}\left(p_N + \sqrt{p_N{}^2 + 4q_N}\right) \Rightarrow q_N = N(N - p_N),\ p_N = 1,2,3,... \qquad (2.17)$$

Here are the calculations:

$$\begin{aligned}
&p_N = 0: \quad q_1 = 1(1-0) = 1; \quad q_4 = 4(4-0) = 6; \quad q_9 = 9(9-0) = 9;,,,\\
&p_N = 1: \quad q_2 = 2(2-1) = 2; \quad q_3 = 3(3-1) = 6; \quad q_4 = 4(4-1) = 12;,,,\\
&p_N = 2: \quad q_3 = 3(3-2) = 3; \quad q_4 = 4(4-2) = 8; \quad q5 = 5(5-2) = 15;,,, \qquad (2.18)\\
&p_N = 3: \quad q_4 = 4(4-3) = 4; \quad q_5 = 5(5-3) = 10; \quad q_6 = 6(6-3) = 18;,,,\\
&...
\end{aligned}$$

3. There is a diagonal $p = q + 1$ (conventionally, the first diagonal) of the integer roots $\Phi(p,q) = N$, whose values begin with number $\Phi(p = 0, q = 1) = N = 1$ and grow in directions of simultaneous increase of the factors p and q.

4. To the right of the first diagonal of the integer values of the roots $\Phi(p,q) = N$, the following diagonals of the integer values of the roots $\Phi(p,q) = N$ with an increasing inclination relative to the first diagonal are formed (see Table 2.5, where the empty cells correspond to the irrational values of the roots).

Table 2.5

The whole values of the roots of the equation $\Phi(p,q)^2 - p\Phi(p,q) - q = 0$
with positive and integer factors p, q

q p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	1			2					3							4								
1		**2**				**3**						**4**								**5**				
2			**3**					**4**							**5**									
3				**4**						**5**								**6**						
4					**5**							**6**									**7**			
5						**6**								**7**										**8**
6							**7**									**8**								
7								**8**										**9**						
8									**9**											**10**				
9										**10**														

5. For a pair of values of factors p and q (cell indices in tables), the irrational roots of the quadratic equation $\Phi(p,q)^2 - p\Phi(p,q) - q = 0$ with integer positive ($p>0$, $q>0$) values p and q equally obey the Vieta`s theorem:

$$\begin{cases} \Phi(p,q) - \varphi(p,q) = p, \\ \Phi(p,q)\cdot\varphi(p,q) = q. \end{cases} \tag{2.19}$$

6. The first vertical column $p = 0,1,2,3,...;\ q = 1$ corresponds to the so-called “Metallic mean” [18; 25, 26; 37, 38]. The roots of the equation $\Phi(p,1)^2 - p\Phi(p,1) - 1 = 0$ are irrational and the Vieta`s theorem becomes:

$$\begin{cases} \Phi(p,1) - \varphi(p,1) = p, \\ \Phi(p,1)\cdot\varphi(p,1) = 1. \end{cases} \tag{2.20}$$

7. A generalized model of proportional division not only includes integer values of coefficients p, q [42; 43], but also non-objective values p, q .

Let us formulate the general equation of the proportional division. The GR $\frac{L}{x} = \frac{x}{L - x}$ gives the quadratic equations

$$x^2 + Lx - L^2 = 0\,(a) \Rightarrow \begin{cases} \frac{L^2}{x^2} - \frac{L}{x} - 1 = 0 \Rightarrow \Phi - \Phi - 1 = 0, (b) \\ \frac{x^2}{L^2} + \frac{x}{L} - 1 = 0 \Rightarrow \varphi + \varphi - 1 = 0, (c) \end{cases} . \tag{2.21}$$

The equation (2.21(a)) relates to the quadratic equation with coefficients L and L^2 unequal in modulus, for which the coordinate of the division is:

$$x_{\pm,L} = L\left(\frac{1 \pm \sqrt{5}}{2}\right) = \begin{cases} x_{+,L} = L\varphi, \\ x_{-,L} = -L\Phi, \end{cases} \tag{2.22}$$

that is, the numbers φ and Φ are the coefficients of the normalization of the coordinates of division $x_{\pm}$ in the proportional division of the whole.

The specificity of the quadratic equations (2.21(b)) and (2.21(c)) is that the coefficients are equal in modulus and in this case they are equal to unity.

In Tables 2.3 and 2.4 the solutions to the equation (2.21(b)) and the equation $\varphi(k) = \frac{k}{\Phi(k)}$ are given as the basic ones, that is, the solutions given the factors $|p| = |q| = k$ equal in modulus. Therefore, the general equation of the proportional division is:

$$\Phi(k) = \frac{L}{x_k} : \Phi(k)^2 - k\Phi(k) - k^2 = 0 \Rightarrow \frac{L}{x_k} = \frac{x_k}{L/k - x_k}. \tag{2.23}$$

Having defined the parameter L from (2.22) and (2.23), we obtain the relation between the coordinates of the proportional divisions x_1 and x_k :

$$\begin{cases} L = x_1(1+\sqrt{5}) \\ L == x_k k(1+\sqrt{1+4/k} \end{cases} \Rightarrow x_1(1+\sqrt{5}) = x_k k(1+\sqrt{1+4/k} \Rightarrow$$
$$\Rightarrow x_1 \frac{1+\sqrt{5}}{k\left(1+\sqrt{1+4/k}\right)} \Rightarrow \begin{cases} x_k\big|_{k \to 1} \to x_1; \\ x_k\big|_{k \to +\infty} \to 0. \end{cases} \tag{2.24}$$

Thus, for $k \geq 1$ the values x_k are localized in the interval $0 \leq x_k \leq x_1$. Table 2.5, similarly to Tables 2.3 and 2.4, presents the results of calculations for non-integer positive values of factors p, q, which prove that the solutions are irrational and the property of the product of roots is fulfilled for them:

$$\Phi(p,q)\cdot\varphi(p,q) = q. \tag{2.25}$$

Thus, the results confirm the idea that it is possible to construct the set of irrational roots of quadratic equations with positive arbitrary values of the factors p, q of a quadratic equation $\Phi(p,q)^2 - p\Phi(p,q) - q = 0$. Let us substantiate the general law of proportion:

$$\begin{gathered}\Phi(p,q)^2 - p\Phi(p,q) - q = 0 \Rightarrow \left(\frac{L}{x(p,q)}\right)^2 - p\frac{L}{x(p,q)} - q = 0 \Rightarrow \\ \Rightarrow \frac{L}{x(p,q)}\left[\frac{L}{x(p,q)} - p\right] = q \Rightarrow \frac{L}{x(p,q)} = \frac{q}{\dfrac{L}{x(p,q)} - p} = \frac{q\cdot x(p,q)}{L - p\cdot x(p,q)} \Rightarrow \\ \Rightarrow \frac{L}{x(p,q)} = \frac{q\cdot x(p,q)}{L - p\cdot x(p,q)} \Rightarrow \text{ if } p = q = k \text{ then } \frac{L}{x(k)} = \frac{k\cdot x(k)}{L - k\cdot x(k)}.\end{gathered} \tag{2.26}$$

Table 2.5

$\Phi(p,q)^2 - p\Phi(p,q) - q = 0$. The factors $p = q$ are positive and non-integer numbers

$q \quad p$	0.76 $\Phi_{p,q}\cdot\varphi_{p,q} = 0.76$	1.6 $\Phi_{p,q}\cdot\varphi_{p,q} = 1.6$	3.14 $\Phi_{p,q}\cdot\varphi_{p,q} = 3.14$	15.3 $\Phi_{p,q}\cdot\varphi_{p,q} = 15.3$
0.76	$\Phi = 1.331$ $\varphi = 0.571$	$\Phi = 1.70$ $\varphi = 0.94$	$\Phi = 2.192$ $\varphi = 1.432$	$\Phi = 4.31$ $\varphi = 3.55$
1.6	$\Phi = 1.983$ $\varphi = 0.383$	$\Phi = 2.297$ $\varphi = 0.697$	$\Phi = 2.744$ $\varphi = 1.144$	$\Phi = 4.792$ $\varphi = 3.192$
3.14	$\Phi = 2.366$ $\varphi = 0.226$	$\Phi = 3.586$ $\varphi = 0.446$	$\Phi = 3.937$ $\varphi = 0.797$	$\Phi = 5.785$ $\varphi = 2.645$

If the sign of the factor p is changed to the opposite $p \leq 0$ ($q \geq 0$), then we obtain the equation $\Phi(p,q)^2 + p\Phi(p,q) - q = 0$, whose solutions will be mirror symmetric to the solutions to the equation $\Phi(p,q)^2 - p\Phi(p,q) - q = 0$. Therefore, let us consider the case $p \geq 0$ $q \leq 0$, that is, the solutions to the quadratic equation $\Phi(p,q)^2 - p\Phi(p,q) + q = 0$, which are given in Table 2.6. In this case, the roots are valid given $p^2 \geq 4q$ (empty cells correspond to invalid solutions). For valid positive the line of their whole values corresponds to the coefficients $p = m; q = m - 1, m = 1, 2, 3, ...$ Other roots are irrational. The analysis of the data in Table 6 shows that for real solutions the analytical relation between $\Phi(k)$, $\varphi(k)$ is valid:

$$\Phi(p,q)^2 - p\Phi(p,q) + q = 0: \begin{cases} p = const: & \Phi(p,q) + \varphi(p,q) = p; \\ q = const: & \Phi(p,q)\cdot\varphi(p,q) = 1. \end{cases} \tag{2.27}$$

Negative roots are equal to $\Phi_{-}(k) = -\varphi(k)$. Let us establish the general law of proportion:

$$\Phi(p,q)^2 - p\Phi(p,q) + q = 0 \Rightarrow \left(\frac{L}{\xi(p,q)}\right)^2 - p\frac{L}{\xi(p,q)} + q = 0 \Rightarrow$$

$$\Rightarrow \frac{L}{\xi(p,q)}\left[\frac{L}{\xi(p,q)} - p\right] = -q \Rightarrow \frac{L}{\xi(p,q)} = \frac{-q}{\frac{L}{\xi(p,q)} - p} = \frac{q\cdot\xi(p,q)}{p\cdot\xi(p,q) - L} \Rightarrow \tag{2.28}$$

$$\Rightarrow \frac{L}{\xi(p,q)} = \frac{q\cdot\xi(p,q)}{p\cdot\xi(p,q) - L}.$$

Table 2.6

$\Phi(p,q)^2 - p\Phi(p,q) + q = 0$. **The factors $p>0,\ q<0$ are integer roots with different signs**

q p	1	2	3	4	5	6	7
1							
2							
3	$\Phi = 2.618$ $\varphi = 0.382$						
4	$\Phi = 3.732$ $\varphi = 0.268$	$\Phi = 3.414$ $\varphi = 0.586$	$\Phi = 3$ $\varphi = 1$				
5	$\Phi = 4.79$ $\varphi = 0.21$	$\Phi = 4.56$ $\varphi = 0.44$	$\Phi = 4.303$ $\varphi = 0.697$	$\Phi = 4$ $\varphi = 1$	$\Phi = 3.62$ $\varphi = 1.38$	$\Phi = 3$ $\varphi = 2$	
6	$\Phi = 5.83$ $\varphi = 0.17$	$\Phi = 5.646$ $\varphi = 0.354$	$\Phi = 5.45$ $\varphi = 0.55$	$\Phi = 5.236$ $\varphi = 0.764$	$\Phi = 5$ $\varphi = 1$	$\Phi = 4.732$ $\varphi = 1.268$	$\Phi = 4.414$ $\varphi = 1.586$
7	$\Phi = 6.854$ $\varphi = 0.146$	$\Phi = 6.7$ $\varphi = 0.3$	$\Phi = 6.542$ $\varphi = 0.458$	$\Phi = 6.37$ $\varphi = 0.63$	$\Phi = 6.2$ $\varphi = 0.8$	$\Phi = 6$ $\varphi = 1$	$\Phi = 5.79$ $\varphi = 1.21$

Table 2.7

$\Phi(p,q)^2 + p\Phi(p,q) + q = 0$. **The factors $p<0,\ q<0$ are negative and integer numbers**

q p	1	2	3	4	5	6	7
1							
2							
3	$\Phi = -2.618$ $\varphi = -0.382$						
4	$\Phi = -3.732$ $\varphi = -0.268$	$\Phi = -3.414$ $\varphi = -0.586$	$\Phi = -3$ $\varphi = -1$				
5	$\Phi = -4.79$ $\varphi = -0.21$	$\Phi = -4.56$ $\varphi = -0.44$	$\Phi = -4.303$ $\varphi = -0.697$	$\Phi = -4$ $\varphi = -1$	$\Phi = -3.62$ $\varphi = -1.38$	$\Phi = -3$ $\varphi = -2$	
6	$\Phi = -5.83$ $\varphi = -0.17$	$\Phi = 5.646$ $\varphi = 0.354$	$\Phi = 5.45$ $\varphi = 0.55$	$\Phi = -5.236$ $\varphi = -0.764$	$\Phi = -5$ $\varphi = -1$	$\Phi = -4.732$ $\varphi = -1.268$	$\Phi = -4.414$ $\varphi = -1.586$
7	$\Phi = -6.854$ $\varphi = -0.146$	$\Phi = -6.7$ $\varphi = -0.3$	$\Phi = -6.542$ $\varphi = -0.458$	$\Phi = -6.37$ $\varphi = -0.63$	$\Phi = -6.2$ $\varphi = -0.8$	$\Phi = -6$ $\varphi = -1$	$\Phi = -5.79$ $\varphi = -1.21$

Given $p<0$ and $q<0$, the quadratic equation $\Phi(p,q)^2+p\Phi(p,q)+q=0$ has imaginary roots given $p^2-4q<0$ (blank cells in Table 2.7). Negative roots in absolute values coincide with the analogous ones in Table 2.6. The general law of proportion in this case is:

$$\Phi(p,q)^2+p\Phi(p,q)+q=0 \Rightarrow \left(\frac{L}{x(p,q)}\right)^2+p\frac{L}{x(p,q)}+q=0 \Rightarrow \frac{L}{x(p,q)}\left[\frac{L}{x(p,q)}+p\right]=$$
$$=-q \Rightarrow \frac{L}{x(p,q)}=\frac{-q}{\dfrac{L}{x(p,q)}+p}=\frac{q\cdot x(p,q)}{p\cdot x(p,q)-L} \Rightarrow \frac{L}{[-x(p,q)]}=\frac{q\cdot[-x(p,q)]}{L-p[-x(p,q)]},\ so\ as\ \Phi<0. \tag{2.29}$$

Let us investigate the correlation of the equations (2.9) with the geometry of Pythagoras, which in Pythagorean representation are:

$$\begin{cases}\Phi(k)^2-k\Phi(k)-k=0,\\ \varphi(k)^2+k\varphi(k)-k=0,\end{cases} \Rightarrow \begin{cases}\Phi(k)^2=\left(\sqrt{k\Phi(k)}\right)^2+\left(\sqrt{k}\right)^2,\\ \left(\sqrt{k}\right)^2=\varphi(k)^2+\left(\sqrt{k\varphi(k)}\right)^2.\end{cases} \tag{2.30}$$

Thus, the Pythagorean triangle will get the value of the proportionality coefficients: for the hypotenuse Φ_k, the height $\sqrt{k\Phi_k}$, the basis $\sqrt{k}$ (Fig. 2.3 (b)) and the height forming angle for different values k will be equal to:

$$\tan\alpha_k=\frac{\sqrt{k\Phi_k}}{\sqrt{k}}=\sqrt{\Phi_k}=\begin{cases}k=0.15:\alpha_2=\sqrt{0.469}=0.601=34.436^0\\ k=1.00:\alpha_1=\sqrt{1.618}=0.905=51.853^0\\ k=2.00:\alpha_2=\sqrt{2.732}=1.027=58.856^0\\ k=3.14:\alpha_2=\sqrt{3.937}=1.104=63.282^0\\ \qquad\qquad\qquad\ldots\end{cases} \tag{2.31}$$

and with the height k it approximates 90^0. The area of the corresponding golden Pythagorean triangles is:

$$S_k=\frac{1}{2}\sqrt{k}\cdot\sqrt{k\Phi_k}=\frac{1}{2}k\cdot\sqrt{\Phi_k}. \tag{2.32}$$

Conclusions

The main conclusions of the work are formulated above, so let's note this. The fact that there is a plurality of irrational values of the roots of a quadratic equation, as the basic proportional division of the whole into two unequal parts, makes it possible to apply optimization algorithms, including micro and nanoelectronic systems, much more efficiently. However, this is the task of a separate study.

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