Numerical Solution of Singularly Perturbed Convection-Diffusion Equations

Mehmet Giyas Sakar[1], Fevzi Erdogan[2]

1. Yuzuncu Yil University, Faculty of Sciences, Department of Mathematics, Van, Turkey, E-mail:giyassakar@hotmail.com

2. Yuzuncu Yil University, Faculty of Sciences, Department of Mathematics, Van, Turkey, E-mail: ferdogan@yyu.edu.tr

***Abstract* – In this paper, a new method is given for solving singularly perturbed convection-diffusion problems. The present method is based on combining the asymptotic expansion method and the variational iteration method (VIM) with an auxiliary parameter. Numerical results show that the present method can provide very accurate numerical solutions not only in the boundary layer, but also away from the layer..**

Keywords – Convection-diffusion problems, boundary layer, variational iteration method, asymptotic expansion, auxiliary parameter..

I. Introduction

In this paper, we consider the following singularly perturbed convection-diffusion problems [1],

$$Lu(x) = \varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x,u),$$
$$0 \le x \le 1 \tag{1}$$

with the boundary conditions

$$u(0) = A,\ u(1) = B \tag{2}$$

where $0 < e << 1$, $a(x)$, $b(x)$ and $f(x,u)$ are assumed to be sufficiently smooth, and such that (1)-(2) has a unique solution. Further, it is assumed that the function $a(x) \ge a > 0$, a is a constant. Under the above assumption, singularly perturbed convection-diffusion problem (1)-(2) possesses a unique smooth solution with boundary layer on the left side of the domain $[0,1]$.

Singularly perturbed problems depend on small positive parameter which multiplying with highest derivative term. This parameter cause to the solution changes quickly in some region and changes slowly in some other regions. The existence of perturbation parameter lead to complication, so classical numerical techniques not useful to solve such problems. Because of this, different techniques are needed to overcome this complication. In the recent times, a wide range of particular methods have been improved by a lot of authors for approximate solution of singularly perturbed problems [2-5].

II. The solution of singularly perturbed convection-diffusion problem Eqs.(1)-(2)

In this section the asymptotic expansion approximation to the solution of singularly perturbed convection-diffusion problem (1)-(2) is constructed.

Theorem 2.1. (Maximum Principle) Suppose v is a smooth function satisfying $v(0) \ge 0$, $v(1) \ge 0$ and $Lv(x) \le 0$ for all $0 \le x \le 1$. Then $v(x) \ge 0$ for all $0 \le x \le 1$.

It follows directly that problem (1)-(2) has a unique solution. Let $u(x)$ and $u_0(x)$ be the solutions of (1)-(2) and its reduced problem, respectively

$$a(x)u_0'(x) + b(x)u_0(x) = f(x,u_0) \tag{3}$$

$$u(1) = B, \tag{4}$$

Then, the zeroth order asymptotic expansion approximation,

$$u_{as} = u_0(x) + v_0(x), \tag{5}$$

where $v_0(x)$ is the solution of the below equations (6)-(7).

$$v_0''(x) + a(0)v_0'(x) = 0,\ x \in (0,\infty) \tag{6}$$

$$v_0(0) = A - u_0(0),\ v_0(\infty) = 0 \tag{7}$$

We see directly $v_0(x)$ is given by

$$v_0(x) = (u(0) - u_0(0))e^{\frac{-a(0)x}{e}}. \tag{8}$$

We note that Eqs. (1)-(2) also has a unique solution but that the solution will not in general satisfy the boundary condition at $x = 0$.

Theorem 2.2. For sufficiently smooth $a(x)$, $b(x)$ and $f(x,u)$, the zeroth order asymptotic expansion approximation $u_{as}(x)$ satisfies the inequality,

$$|u_{as} - u(x)| < e, \tag{9}$$

where $u(x)$ is the solution of (1)-(2). The proof of theorem 2.1. and theorem 2.2. can be found [6]. In order to obtain zeroth order asymptotic expansion approximation $u_{as}(x)$, it remains only obtain the solution $u_0(x)$ of terminal value problem (3)-(4).

III. The solution of terminal value problem (3)-(4)

The solution of terminal value problem of (3)-(4) will be obtained by using the variational iteration method with an auxiliary parameter. Terminal value problem (3)-(4) can be converted into the following equivalent form

$$w'(x) = \frac{f(x,w)}{a(x)} - \frac{b(x)}{a(x)}w(x) = F(x,w), \tag{10}$$
$$0 \le x \le 1$$

$$w(x) = B \tag{11}$$

for (10), according to the VIM, we can obtain the following iteration formula:

$$w_{n+1}(s) = w_0(s) + \int_1^x F(s, w_n(s))ds, \tag{12}$$

where $w_0 = B.$

IV. Variational iteration method with an auxiliary parameter

In this section, the basic ideas of the variational iteration method [7] are introduced. We consider the following differential equations:

$$Tu = Lu + Ru + Nu - g(x) = 0 \qquad (13)$$

where L is the highest order derivative that is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms and $g(x)$ is a inhomogeneous term. According to variational iteration method, we can write down a correction functional as follows :

$$u_{n+1}(x) = u_n(x) + \int_0^x l(x,t)Tu_n(t)dt, \qquad (14)$$

An unknown auxiliary parameter $\mathbf{h}$ can be inserted into the variational iteration algorithm (14),

$$u_1(x,\mathbf{h}) = u_0(x) + \mathbf{h}\int_0^x l(x,t)Tu_n(t)dt, \qquad (15)$$

$$u_{n+1}(x,\mathbf{h}) = u_n(x,\mathbf{h}) + \mathbf{h}\int_0^x l(x,t)Tu_n(t)dt,\ n \geq 1 \qquad (16)$$

where $l(x,t)$ is a general lagrange multiplier which can be optimally identified via variational theory [8]. The approximate solutions $u_n(x,h)$, $n \geq 1$ contains the auxiliary parameter $\mathbf{h}$. By means of the so-called $\mathbf{h}$-curve, it is straightforward to choose a proper value of $\mathbf{h}$ which ensures that the approximate solutions are convergent [9].

Finally, we approximate the solution $u(x) = \lim_{n\to\infty} u_n(x,\mathbf{h})$ by the nth term $u_n(x,\mathbf{h})$. The above series solutions generally converge very rapidly. For the convergence of the variational iteration method, we will give the following theorem.

As given by [10], at the nth-order of approximation, one can define the exact square residual error

$$\Delta_n = \int_\Omega [Tu_n(x)]^2 dx \qquad (17)$$

However, it is proven by [10] that the exact residual error Δ_n defined by equation (17) needs too much CPU time to calculate even if the order of approximation is not very high. Thus, to greatly decrease the CPU time, we use here the so-called averaged square residual error $\sqrt{E_n}$ defined by

$$\sqrt{E_n} = \left(\frac{1}{m+1} \sum_{j=0}^{m} \left(T\left[u_n\left(\frac{j}{m}, \mathbf{h} \right) \right] \right)^2 \right)^{1/2} \qquad (18)$$

Theorem 3.1. (Banach's Fixed Point Theorem). Assume that BS is a Banach space and

$$A : BS \to BS$$

is a nonlinear mapping, and suppose that

$$\left\| A[u] - A[v] \right\| \leq a \left\| u - v \right\|,\ u, v \in BS$$

for some constants $a < 1$. Then A has a unique fixed point. Furthermore, the sequence

$$u_{n+1} = A[u_n],$$

with an arbitrary choice of $u_0 \in BS$, converges to the fixed point of A.

According to Theorem 3.1. for the nonlinear mapping

$$A[u(x)] = u(x) + \mathbf{h}\int_0^x l(x,t) \left\{ \begin{aligned} &Lu(t) + Ru(t) \\ &+ Nu(t) - g(t) \end{aligned} \right\} dt,$$

a sufficient condition for convergence of the variational iteration method with an auxiliary parameter is strictly a contraction of A. Furthermore, the sequence (19) converges to the fixed point of A which is also the solution of problem (10).

V. Numerical Example

In this section, we apply the variational iteration method which presented in previous sections to singularly perturbed convection-diffusion problems.

Example 5.1. We consider the following singularly perturbed linear convection-diffusion problems [2]:

$$eu''(x) + u'(x) = 1 + 2x,\ 0 < x < 1, \qquad (20)$$

$$u(0) = 0,\ u(1) = 1. \qquad (21)$$

It is easy to see that its exact solution is:

$$u(x) = x(x + 1 - 2e) + \frac{(2e-1)\left(1 - e^{-\frac{x}{e}}\right)}{1 - e^{-\frac{1}{e}}} \qquad (22)$$

The exact solutions and numerical results for different e values are given in Table 1-2. Optimal $\mathbf{h}$ parameter values are given in Table 3 with $n = 3, m = 100$.

TABLE 1

EXACT SOLUTIONS FOR EXAMPLE 5.1 WITH $n = 3$

x	$e = 10^{-3}$	$e = 10^{-4}$	$e = 10^{-5}$
0.0001	-0.0948724468	-0.6318941447	-0.9998345930
0.0005	-0.3921831516	-0.9925632506	-0.9994797600
0.001	-0.6298573177	-0.9987538092	-0.9989790200
0.005	-0.9862605289	-0.9947760000	-0.9949551000
0.1	-0.8882000000	-0.8898200000	-0.8899820000
0.3	-0.6086000000	-0.6098600000	-0.6099860000
0.5	-0.2490000000	-0.2499000000	-0.2499900000
0.7	0.1906000000	0.1900600000	0.1900060000
0.9	0.7102000000	0.7100200000	0.7100020000

TABLE 2

NUMERICAL RESULTS FOR EXAMPLE 5.1 WITH $n = 3$

x	$e = 10^{-3}$	$e = 10^{-4}$	$e = 10^{-5}$
0.0001	-0.09488739680	-0.6319040435	-0.9998361492
0.0005	-0.39224487210	-0.9925787679	-0.9994813120
0.001	-0.62995626190	-0.9987693775	-0.9989805666
0.005	-0.98641232330	-0.9947911399	-0.9949566037
0.1	-0.88825953030	-0.8898258146	-0.8899825717
0.3	-0.60851737770	-0.6098516198	-0.6099851537
0.5	-0.24884889560	-0.2498847980	-0.2499884734
0.7	0.19074591560	0.1900746508	0.1900074692
0.9	0.71026705620	0.7100267268	0.7100026740

TABLE 3

OPTIMAL $\mathbf{h}$ VALUES FOR DIFFERENT e WITH $n = 3$

	$e = 10^{-3}$	$e = 10^{-4}$	$e = 10^{-5}$
$\mathbf{h}$	0.90271	0.95483	0.97903

Conclusion

In this paper, we have demonstrated the suitability of asymptotic expansion and the variational iteration method with an auxiliary parameter for solving singularly perturbed convection-diffusion problems. The hybrid method can decrease a number of computation. Numerical results show that the hybrid method is suitable and very effective.

References

[1] F. Geng, M. Cui, Analytical approximation to solutions of singularly perturbed boundary value prob- lems, Bull. Malays. Math. Sci. Soc. 33 (2) (2010) 221-232.

[2] Y. N. Reddy, P. P. Chakravarthy, Numerical patching method for singularly perturbed two-point boundary value problems using cubic splines, Applied Mathematics and Computation 149 (2004) 441-468.

[3] M. K. Kadalbajoo, A. S. Yadaw, D. Kumar, Comparative study of singularly perturbed two-point BVPs via: Fitted-mesh finite difference method, B-spline collocation method and finite element method, Applied Mathematics and Computation 204 (2) (2008) 713-725.

[4] P. Zhu, S. Xie, Higher order uniformly convergent continuous/discontinuous Galerkin methods for singularly perturbed problems of convection-diffusion type, Applied Numerical Mathematics 76 (2014) 48-59.

[5] M. Brdar, H. Zarin, A singularly perturbed problem with two parameters on a Bakhvalov-type mesh, Journal of Computational and Applied Mathematics 292 (2016) 307-319.

[6] E. P. Doolan, J. J. H. Miller, W. H. A. Schilders, Uniform numerical methods for problems with initial and boundary layers, Boole Press, Dublin, 1980.

[7] J. H. He, Variational iteration method for delay differential equations, Commun. Nonlinear Sci. Numer. Simul., 2 (4) (1997) 235-236.

[8] M. Inokuti, H. Sekine, T. Mura, General use of the Lagrange multiplier in nonlinear mathematical physics, in:S.Nemat-Nasser(Ed.), Variational Method in the Mechanics of Solids, Pergamon Press, New York, 1978, pp. 156-162.

[9] M. G. Sakar, O. Saldır, Improving variational iteration method with auxiliary parameter for nonlinear time-fractional partial differential equations, Journal of Optimization Theory and Applications 174 (2) (2017) 530-549.

[10] Liao, S. J., An optimal homotopy-analysis approach for strongly nonlinear differential equations, Commun. Nonlinear Sci. Numer. Simulat. 15 2003-2016 (2010).