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ON ASYMPTOTIC PROPERTIES OF ENTIRE FUNCTIONS, SIMILAR TO THE ENTIRE FUNCTIONS OF COMPLETELY REGULAR GROWTH

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Let f be an entire function of order $\rho \in (0;+\infty)$ with the indicator h and let for some $\rho_1 \in (0;\rho)$ there exists an exceptional set $U \subset \mathbb{C}$ such that $\log|f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_1})$, $U \not\ni z = re^{i\varphi} \to \infty$, and U can be covered by a system of pairwise disjoint disks $U_k = \{z : |z-a_k| < \tau_k\}$, $k \in \mathbb{N}$, satisfying $\sum_{k\in\mathbb{N}} \tau_k < +\infty$, $\sum_{k\in\mathbb{N}} \tau_k |\log \tau_k| < +\infty$. Then there exists $\rho_2 \in (0;\rho)$ such that $\int_1^r t^{-1} \log|f(te^{i\varphi})|\,dt = \frac{r^\rho}{\rho} h(\varphi) + o(r^{\rho_2})$ as $r \to +\infty$ uniformly with respect to $\varphi \in [0;2\pi]$.

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Introduction and main result

One of the main theorems of the theory of entire functions of completely regular growth in the sense of Levin-Pflüger (see [1, pp. 182–217], [2]) is

Levin's Theorem. ([1, p. 197]) *In order that an entire function f of order $\rho \in (0;+\infty)$ with the indicator h be of completely regular growth, it is necessary and sufficient that for every $\varphi \in [0;2\pi]$ there exists one of the following limits:*

$$\lim_{r\to+\infty} r^{-\rho} \int_1^r \frac{\log|f(te^{i\varphi})|}{t}\,dt = \frac{1}{\rho} h(\varphi);$$

$$\lim_{r\to+\infty} r^{-\rho} \int_1^r \frac{dt}{t} \int_1^t \frac{\log|f(ue^{i\varphi})|}{u}\,du = \frac{1}{\rho^2} h(\varphi).$$

Similar results for entire functions of ρ-regular growth were obtained by A. F. Grishin [3] and for meromorphic functions of completely regular growth of finite λ-type ([4, p. 75]) by A. A. Kondratyuk [4, p. 112], Ya. V. Vasyl'kiv [5] and others.

In [6,7] (see also [8]) a new notion of an entire function of improved regular growth was introduced and a criteria for this regularity in the sense of zero distribution were established when the zeros are located on a finite number of rays.

Definition. ([6], [8, p. 34]) *We say that an entire function f is of improved regular growth, if for some $\rho \in (0;+\infty)$, $\rho_1 \in (0;\rho)$, and some 2π-periodic ρ-trigonometrically convex function $h(\varphi) \not\equiv -\infty$ there exists an exceptional set $U \subset \mathbb{C}$ such that*

$$\log|f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_1}),\ U \not\ni z = re^{i\varphi} \to \infty, \quad (1)$$

and U can be covered by a system of disks with finite sum of radii.

Using a method different from that in ([1, pp. 188–194]) we obtain the following statement, which connected with the study of entire functions of improved regular growth in general case (i.e., with zeros on arbitrary system of rays).

Theorem 1. *Let f be an entire function of order $\rho \in (0;+\infty)$ with the indicator h and let for some $\rho_1 \in (0;\rho)$ there exists an exceptional set $U \subset \mathbb{C}$ such that relation (1) holds and U can be covered by a system of pairwise disjoint disks $U_k = \{z : |z-a_k| < \tau_k\}$, $k \in \mathbb{N}$, satisfying*

$$\sum_{k\in\mathbb{N}} \tau_k < +\infty, \quad \sum_{k\in\mathbb{N}} \tau_k |\log \tau_k| < +\infty. \quad (2)$$

Then there exists $\rho_2 \in (0;\rho)$ such that

$$J_f^r(\varphi) := \int_1^r \frac{\log|f(te^{i\varphi})|}{t}\,dt = \frac{r^\rho}{\rho} h(\varphi) + o(r^{\rho_2}) \quad (3)$$

as $r \to +\infty$ uniformly with respect to $\varphi \in [0;2\pi]$.

Remark that, if the function f satisfies the conditions of Theorem 1, then it an entire function of improved regular growth.

Auxiliary Lemmas

To prove Theorem 1, we need two lemmas.

Lemma 1. [6], [8, p. 52]) *Let $\rho \in (0;+\infty)$. If f is an entire function of improved regular growth, then the asymptotic inequality*

$$\log|f(z)| \le |z|^\rho h(\varphi) + o(|z|^{\rho_3}), \quad z = re^{i\varphi} \to \infty,$$

holds for some $\rho_3 \in (0;\rho)$.

Lemma 2. ([?], [?, p. 55]) *Let $\rho \in (0;+\infty)$, $\rho_1 \in (0;\rho)$ and f is an entire function of improved regular growth. Then there exists a sequence (r_s) such that*

$$0 < r_s \uparrow +\infty, \quad r_{s+1}^\rho - r_s^\rho = o(r_s^{\rho_1}), \quad s \to +\infty,$$

and

$$\log|f(r_s e^{i\varphi})| = r_s^\rho h(\varphi) + o(r_s^{\rho_1}), \quad s \to +\infty, \tag{4}$$

uniformly with respect to $\varphi \in [0;2\pi]$.

Proof of Theorem 1

Taking into account Lemma 1, we get

$$J_f^r(\varphi) \le \frac{r^\rho}{\rho} h(\varphi) + o(r^{\rho_3}), \quad r \to +\infty, \tag{5}$$

for some $\rho_3 \in (0;\rho)$ uniformly with respect to $\varphi \in [0;2\pi]$. Let us prove that

$$J_f^r(\varphi) \ge \frac{r^\rho}{\rho} h(\varphi) + o(r^{\rho_3}), \quad r \to +\infty, \tag{6}$$

uniformly with respect to $\varphi \in [0;2\pi]$. Let $U^* = \bigcup_{k\in\mathbb{N}} U_k^*$, where $U_k^* = [|a_k| - \tau_k; |a_k| + \tau_k]$. We have

$$J_f^r(\varphi) = \left(\int\limits_{[1;r]\setminus U^*} + \int\limits_{[1;r]\cap U^*} \right) \frac{\log\left|f(te^{i\varphi})\right|}{t}\, dt$$

$$\ge \int\limits_{[1;r]\setminus U^*} (t^{\rho-1}h(\varphi) + o(t^{\rho_1 - 1}))\, dt$$

$$+ \int\limits_{[1;r]\cap U^*} \frac{\log\left|f(te^{i\varphi})\right|}{t}\, dt$$

$$= h(\varphi) \int\limits_{[1;r]} t^{\rho-1}\, dt - h(\varphi) \int\limits_{[1;r]\cap U^*} t^{\rho-1}\, dt$$

$$+ \int\limits_{[1;r]\cap U^*} \frac{\log\left|f(te^{i\varphi})\right|}{t}\, dt + o(r^{\rho_1})$$

$$\ge \frac{r^\rho}{\rho} h(\varphi) - |h(\varphi)| \cdot \begin{cases} r^{\rho-1} \int_{U^*} dt, & \rho \ge 1, \\ \int_{U^*} dt, & \rho < 1, \end{cases}$$

$$+ \int\limits_{[1;r]\cap U^*} \frac{\log\left|f(te^{i\varphi})\right|}{t}\, dt + o(r^{\rho_1}) \ge \frac{r^\rho}{\rho} h(\varphi)$$

$$+ \int\limits_{[1;r]\cap U^*} \frac{\log\left|f(te^{i\varphi})\right|}{t}\, dt + o(r^{\rho_4}), \quad r \to +\infty, \quad \rho_4 < \rho. \tag{7}$$

Now we shall show that there exists $\rho_5 \in (0;\rho)$ such that

$$\int\limits_{[1;r]\cap U^*} \frac{\log\left|f(te^{i\varphi})\right|}{t}\, dt = o(r^{\rho_5}), \quad r \to +\infty. \tag{8}$$

Let $r \in [|a_\nu| - \tau_\nu; |a_\nu| + \tau_\nu)$. Then

$$\int\limits_{[1;r]\cap U^*} \frac{\log\left|f(te^{i\varphi})\right|}{t}\, dt = O(1) + \sum_{k=1}^{\nu-1} I(k) + \Omega(\nu), \tag{9}$$

where

$$I(k) := \int\limits_{U_k^*} \frac{\log\left|f(te^{i\varphi})\right|}{t}\, dt, \quad 1 \le k \le \nu - 1,$$

$$\Omega(\nu) := \int\limits_{|a_\nu| - \tau_\nu}^{r} \frac{\log\left|f(te^{i\varphi})\right|}{t}\, dt.$$

For the function $f \not\equiv 0$ in a disk $\{z : |z| < R\}$ takes place the Poisson-Jensen formula ([9, p. 16])

$$\log|f(z)| = \frac{1}{2\pi} \int\limits_0^{2\pi} \log|f(Re^{i\theta})| \mathrm{Re} \frac{Re^{i\theta} + z}{Re^{i\theta} - z}\, d\theta$$

$$+ \sum_{|\lambda_n| < R} \log\left| \frac{R(z - \lambda_n)}{R^2 - z\overline{\lambda}_n} \right|, \quad z = te^{i\varphi}, \tag{10}$$

where $\lambda_n = |\lambda_n| e^{i\varphi_n}$ are the zeros of the function f. Let $k \in \mathbb{N}$ and $m \in \mathbb{N}$ are the numbers such that $r_{i_k} \le |a_k| < r_{i_k+1}$ and $r_m \le 2r_{i_k} < r_{m+1}$, where (r_s) is a sequence from Lemma 2. Put $R = r_{m+2}$ in (10). Using Fubini's theorem, and formulas (4) and (10), we obtain

$$I(k) = \int\limits_{U_k^*} \left(\frac{1}{2\pi} \int\limits_0^{2\pi} \log\left|f(r_{m+2}e^{i\theta})\right| \mathrm{Re} \frac{r_{m+2}e^{i\theta} + te^{i\varphi}}{r_{m+2}e^{i\theta} - te^{i\varphi}}\, d\theta \right.$$

$$\left. + \sum_{|\lambda_n| < r_{m+2}} \log\left| \frac{r_{m+2}(te^{i\varphi} - \lambda_n)}{r_{m+2}^2 - te^{i\varphi}\overline{\lambda}_n} \right| \right) \frac{dt}{t}$$

$$\ge \frac{1}{2\pi} \int\limits_0^{2\pi} (r_{m+2}^\rho h(\theta) + o(r_{m+2}^{\rho_1})) I_1(k)\, d\theta$$

$$+ \sum_{|\lambda_n| < r_{m+2}} I_2(k), \tag{11}$$

where

$$I_1(k) = \int\limits_{U_k^*} \frac{r_{m+2}^2 - t^2}{r_{m+2}^2 - 2tr_{m+2}\cos(\varphi - \theta) + t^2} \frac{dt}{t},$$

$$I_2(k)=\int\limits_{U_k^*}\log\left|\frac{r_{m+2}(te^{i\varphi}-\lambda_n)}{r_{m+2}^2-te^{i\varphi}\overline{\lambda}_n}\right|\frac{dt}{t}.$$

Similarly,

$$\Omega(\nu)\geq\frac{1}{2\pi}\int\limits_0^{2\pi}(r_{m+2}^{\rho}h(\theta)+o(r_{m+2}^{\rho_1}))\Omega_1(\nu)\,d\theta$$
$$+\sum_{|\lambda_n|<r_{m+2}}\Omega_2(\nu), \tag{12}$$

where

$$\Omega_1(\nu)=\int\limits_{|a_\nu|-\tau_\nu}^{r}\frac{r_{m+2}^2-t^2}{r_{m+2}^2-2tr_{m+2}\cos(\varphi-\theta)+t^2}\frac{dt}{t},$$

$$\Omega_2(\nu)=\int\limits_{|a_\nu|-\tau_\nu}^{r}\log\left|\frac{r_{m+2}(te^{i\varphi}-\lambda_n)}{r_{m+2}^2-te^{i\varphi}\overline{\lambda}_n}\right|\frac{dt}{t}.$$

Let us estimate $I_1(k)$, $I_2(k)$, $\Omega_1(\nu)$ and $\Omega_2(\nu)$. We first estimate $I_1(k)$ and $\Omega_1(\nu)$. Using the Lagrange theorem, we get

$$|I_1(k)|\leq\frac{2\tau_k}{|a_k|-\tau_k}+\frac{4\tau_k}{r_{m+2}-(|a_k|+\tau_k)}, \tag{13}$$

and, similarly,

$$|\Omega_1(\nu)|\leq\frac{2\tau_\nu}{|a_\nu|-\tau_\nu}+\frac{4\tau_\nu}{r_{m+2}-(|a_\nu|+\tau_\nu)}. \tag{14}$$

Now we estimate $I_2(k)$. Since

$$\left|\frac{r_{m+2}(te^{i\varphi}-\lambda_n)}{r_{m+2}^2-te^{i\varphi}\overline{\lambda}_n}\right|$$
$$=\sqrt{\frac{r_{m+2}^2(t^2-2t|\lambda_n|\cos(\varphi-\varphi_n)+|\lambda_n|^2)}{r_{m+2}^4-2r_{m+2}^2t|\lambda_n|\cos(\varphi-\varphi_n)+t^2|\lambda_n|^2}},$$

then

$$I_2(k)$$
$$=\frac{1}{2}\int\limits_{U_k^*}\log\frac{r_{m+2}^2(t^2-2t|\lambda_n|\cos(\varphi-\varphi_n)+|\lambda_n|^2)}{r_{m+2}^4-2r_{m+2}^2t|\lambda_n|\cos(\varphi-\varphi_n)+t^2|\lambda_n|^2}\frac{dt}{t}$$
$$\geq\frac{1}{2(|a_k|-\tau_k)}\int\limits_{U_k^*}\log\left(1-\frac{(r_{m+2}^2-t^2)(r_{m+2}^2-|\lambda_n|^2)}{(r_{m+2}^2-t|\lambda_n|)^2}\right)dt$$
$$=\frac{1}{2(|a_k|-\tau_k)}\int\limits_{U_k^*}\big(2\log r_{m+2}+\log(t-|\lambda_n|)^2$$
$$-2\log(r_{m+2}^2-t|\lambda_n|)\big)\,dt$$
$$=\frac{2\tau_k}{|a_k|-\tau_k}\log r_{m+2}+\frac{1}{2(|a_k|-\tau_k)}\int\limits_{U_k^*}\log(t-|\lambda_n|)^2\,dt$$
$$+\log(r_{m+2}^2-(|a_k|-\tau_k)|\lambda_n|)$$
$$-\frac{|a_k|+\tau_k}{|a_k|-\tau_k}\log(r_{m+2}^2-(|a_k|+\tau_k)|\lambda_n|)$$
$$+\frac{r_{m+2}^2}{|\lambda_n|(|a_k|-\tau_k)}\log\frac{r_{m+2}^2-(|a_k|+\tau_k)|\lambda_n|}{r_{m+2}^2-(|a_k|-\tau_k)|\lambda_n|}+\frac{2\tau_k}{|a_k|-\tau_k}$$
$$\geq\frac{2\tau_k}{|a_k|-\tau_k}\log r_{m+2}+\frac{2\tau_k}{|a_k|-\tau_k}$$
$$+\frac{1}{2(|a_k|-\tau_k)}\int\limits_{U_k^*}\log(t-|\lambda_n|)^2\,dt$$
$$-\frac{2\tau_k}{|a_k|-\tau_k}\log(r_{m+2}^2-(|a_k|+\tau_k)|\lambda_n|)$$
$$+\frac{r_{m+2}^2}{|\lambda_n|(|a_k|-\tau_k)}\log\frac{r_{m+2}^2-(|a_k|+\tau_k)|\lambda_n|}{r_{m+2}^2-(|a_k|-\tau_k)|\lambda_n|}. \tag{15}$$

Further, applying Lemma 7.2 from [9, p. 56], we get

$$\int\limits_{U_k^*}\log(t-|\lambda_n|)^2\,dt=\int\limits_{|a_k|-\tau_k}^{|a_k|+\tau_k}\log(t-|\lambda_n|)^2\,dt$$
$$=\int\limits_{|a_k|-\tau_k-|\lambda_n|}^{|a_k|+\tau_k-|\lambda_n|}\log u^2\,du\geq2\int\limits_0^{\tau_k}\log u^2\,du$$
$$=4\tau_k(\log\tau_k-1). \tag{16}$$

Furthermore, using the Lagrange theorem, we obtain

$$\log\frac{r_{m+2}^2-(|a_k|+\tau_k)|\lambda_n|}{r_{m+2}^2-(|a_k|-\tau_k)|\lambda_n|}$$
$$\geq-\frac{2\tau_k|\lambda_n|}{r_{m+2}^2-(|a_k|+\tau_k)|\lambda_n|}. \tag{17}$$

Hence, (15) together with (16) and (17) gives

$$I_2(k)\geq\frac{2\tau_k}{|a_k|-\tau_k}\log r_{m+2}$$
$$-\frac{2\tau_k}{|a_k|-\tau_k}\log(r_{m+2}^2-(|a_k|+\tau_k)|\lambda_n|)+\frac{2\tau_k}{|a_k|-\tau_k}\log\tau_k$$
$$-\frac{2\tau_k}{|a_k|-\tau_k}\cdot\frac{r_{m+2}^2}{r_{m+2}^2-(|a_k|+\tau_k)|\lambda_n|}. \tag{18}$$

Similarly, as above,

$$\Omega_2(\nu)=\frac{1}{2}\int\limits_{|a_\nu|-\tau_\nu}^{r}\omega(t)\frac{dt}{t}\geq\frac{1}{2}\int\limits_{|a_\nu|-\tau_\nu}^{|a_\nu|+\tau_\nu}\omega(t)\frac{dt}{t}$$
$$\geq\frac{2\tau_\nu}{|a_\nu|-\tau_\nu}\log r_{m+2}+\frac{2\tau_\nu}{|a_\nu|-\tau_\nu}\log\tau_\nu$$
$$-\frac{2\tau_\nu}{|a_\nu|-\tau_\nu}\log(r_{m+2}^2-(|a_\nu|+\tau_\nu)|\lambda_n|)$$
$$-\frac{2\tau_\nu}{|a_\nu|-\tau_\nu}\cdot\frac{r_{m+2}^2}{r_{m+2}^2-(|a_\nu|+\tau_\nu)|\lambda_n|}, \tag{19}$$

where

$$\omega(t)$$
$$=\log\left(1-\frac{(r_{m+2}^2-t^2)(r_{m+2}^2-|\lambda_n|^2)}{r_{m+2}^4-2r_{m+2}^2t|\lambda_n|\cos(\varphi-\varphi_n)+t^2|\lambda_n|^2}\right).$$

For this reason, combining (11), (13) and (18), we get

$$I(k) \geq -\frac{1}{2\pi}\left(\frac{2\tau_k}{|a_k|-\tau_k}+\frac{4\tau_k}{r_{m+2}-(|a_k|+\tau_k)}\right)\times$$

$$\times\int_0^{2\pi}(r_{m+2}^{\rho}|h(\theta)|+|o(r_{m+2}^{\rho_1})|)\,d\theta$$

$$+\sum_{|\lambda_n|<r_{m+2}}\left(\frac{2\tau_k}{|a_k|-\tau_k}\log r_{m+2}+\frac{2\tau_k}{|a_k|-\tau_k}\log\tau_k\right.$$

$$-\frac{2\tau_k}{|a_k|-\tau_k}\log(r_{m+2}^2-(|a_k|+\tau_k)|\lambda_n|)$$

$$\left.-\frac{2\tau_k}{|a_k|-\tau_k}\cdot\frac{r_{m+2}^2}{r_{m+2}^2-(|a_k|+\tau_k)|\lambda_n|}\right)$$

$$\geq-\left(\frac{2\tau_k}{|a_k|-\tau_k}+\frac{4\tau_k}{r_{m+2}-(|a_k|+\tau_k)}\right)O(r_{m+2}^{\rho})$$

$$+\sum_{|\lambda_n|<r_{m+2}}\left(\frac{2\tau_k}{|a_k|-\tau_k}\log\tau_k-\frac{2\tau_k}{|a_k|-\tau_k}\log r_{m+2}\right.$$

$$\left.-\frac{2\tau_k}{|a_k|-\tau_k}\cdot\frac{r_{m+2}}{r_{m+2}-(|a_k|+\tau_k)}\right)$$

$$\geq-\left(\frac{2\tau_k}{|a_k|-\tau_k}+\frac{4\tau_k}{r_{m+2}-(|a_k|+\tau_k)}\right)O(r_{m+2}^{\rho})$$

$$-\sum_{|\lambda_n|<r_{m+2}}\left(\frac{2\tau_k}{|a_k|-\tau_k}|\log\tau_k|+\frac{2\tau_k}{|a_k|-\tau_k}\log r_{m+2}\right.$$

$$\left.+\frac{2\tau_k}{|a_k|-\tau_k}\cdot\frac{r_{m+2}}{r_{m+2}-(|a_k|+\tau_k)}\right)$$

$$=-\left(\frac{2\tau_k}{|a_k|-\tau_k}+\frac{4\tau_k}{r_{m+2}-(|a_k|+\tau_k)}\right)O(r_{m+2}^{\rho})$$

$$-n(r_{m+2})\left(\frac{2\tau_k}{|a_k|-\tau_k}|\log\tau_k|+\frac{2\tau_k}{|a_k|-\tau_k}\log r_{m+2}\right.$$

$$\left.+\frac{2\tau_k}{|a_k|-\tau_k}\cdot\frac{r_{m+2}}{r_{m+2}-(|a_k|+\tau_k)}\right),\qquad (20)$$

as $m\to+\infty$, where $n(t)$ is the number of zeros of the function f from the disk $\{z:|z|\leq t\}$.

Similarly, combining (12), (14) and (19), as $m\to+\infty$, we obtain

$$\Omega(\nu)\geq-\left(\frac{2\tau_\nu}{|a_\nu|-\tau_\nu}+\frac{4\tau_\nu}{r_{m+2}-(|a_\nu|+\tau_\nu)}\right)O(r_{m+2}^{\rho})$$

$$-n(r_{m+2})\left(\frac{2\tau_\nu}{|a_\nu|-\tau_\nu}|\log\tau_\nu|+\frac{2\tau_\nu}{|a_\nu|-\tau_\nu}\log r_{m+2}\right.$$

$$\left.+\frac{2\tau_\nu}{|a_\nu|-\tau_\nu}\cdot\frac{r_{m+2}}{r_{m+2}-(|a_\nu|+\tau_\nu)}\right).\qquad (21)$$

Since f is an entire function of normal type with respect to the finite order ρ, then [?] $n(t)\leq O(t^{\rho})$ as $t\to+\infty$. We may assume that $\frac{1}{2}r_{i_k}\leq|a_k|-\tau_k$ and $|a_k|+\tau_k<\frac{3}{2}r_{i_k+1}$ for all $k\in\mathbb{N}$. Finally, summing (2), (9), (20) and (21), we obtain that there exists $\rho_5\in(0;\rho)$ such that

$$\int_{[1;r]\cap U^*}\frac{\log|f(te^{i\varphi})|}{t}\,dt\geq O(1)$$

$$-\sum_{k=1}^{\nu-1}O(r_{m+2}^{\rho})\left(\frac{2\tau_k}{|a_k|-\tau_k}+\frac{2\tau_k}{|a_k|-\tau_k}|\log\tau_k|\right.$$

$$+\frac{2\tau_k}{|a_k|-\tau_k}\log r_{m+2}+\frac{4\tau_k}{r_{m+2}-(|a_k|+\tau_k)}$$

$$\left.+\frac{2\tau_k}{|a_k|-\tau_k}\cdot\frac{r_{m+2}}{r_{m+2}-(|a_k|+\tau_k)}\right)$$

$$-O(r_{m+2}^{\rho})\left(\frac{2\tau_\nu}{|a_\nu|-\tau_\nu}+\frac{2\tau_\nu}{|a_\nu|-\tau_\nu}|\log\tau_\nu|\right.$$

$$+\frac{2\tau_\nu}{|a_\nu|-\tau_\nu}\log r_{m+2}+\frac{4\tau_\nu}{r_{m+2}-(|a_\nu|+\tau_\nu)}$$

$$\left.+\frac{2\tau_\nu}{|a_\nu|-\tau_\nu}\cdot\frac{r_{m+2}}{r_{m+2}-(|a_\nu|+\tau_\nu)}\right)$$

$$\geq-\sum_{k=1}^{\nu-1}O(r_{i_k}^{\rho-1})(\tau_k+\tau_k|\log\tau_k|)+o(r^{\rho_6})$$

$$\geq-O(r^{\rho-1})\sum_{k=1}^{\nu-1}(\tau_k+\tau_k|\log\tau_k|)+o(r^{\rho_6})$$

$$\geq-o(r^{\rho_7})\sum_{k=1}^{\infty}(\tau_k+\tau_k|\log\tau_k|)+o(r^{\rho_6})\geq o(r^{\rho_5})$$

as $r\to+\infty$. Hence, the relation (8) is valid. In view of this, condition (6) follows directly from (7). Thus the inequalities (5) and (6) imply (3). Theorem 1 is proved.

Corollary. *Let the hypotheses of Theorem 1 be satisfied. Then for some $\rho_2\in(0;\rho)$*

$$\int_1^r J_f^t(\varphi)\frac{dt}{t}=\frac{r^{\rho}}{\rho^2}h(\varphi)+o(r^{\rho_2}),\quad r\to+\infty,$$

uniformly with respect to $\varphi\in[0;2\pi]$.

Remark. *We don't know, whether second of conditions (2) is possible to omit in the Theorem 1.*

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ОБ АСИМПТОТИЧЕСКИХ СВОЙСТВАХ ЦЕЛЫХ ФУНКЦИЙ, БЛИЗКИХ К ЦЕЛЫМ ФУНКЦИЯМ ВПОЛНЕ РЕГУЛЯРНОГО РОСТА

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Если для целой функции f порядка $\rho \in (0;+\infty)$ с индикатором h для некоторого $\rho_1 \in (0;\rho)$ существует множество $U \subset \mathbb{C}$, которое содержится в обьединении попарно непересекающихся кругов $U_k = \{z : |z-a_k| < \tau_k\}$, $k \in \mathbb{N}$, таких, что $\sum_{k\in\mathbb{N}} \tau_k < +\infty$, $\sum_{k\in\mathbb{N}} \tau_k |\log \tau_k| < +\infty$, и $\log|f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_1})$, $U \not\ni z = re^{i\varphi} \to \infty$, то для некоторого $\rho_2 \in (0;\rho)$ равномерно по $\varphi \in [0;2\pi]$ выполняется $\int_1^r t^{-1} \log|f(te^{i\varphi})|\,dt = \frac{r^\rho}{\rho} h(\varphi) + o(r^{\rho_2})$, $r \to +\infty$.

Ключевые слова: целая функция улучшеного регулярного возростания, исключительное множество, ассимптотика.

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ПРО АСИМПТОТИЧНІ ВЛАСТИВОСТІ ЦІЛИХ ФУНКЦІЙ, ПОДІБНИХ ДО ЦІЛИХ ФУНКЦІЙ ЦІЛКОМ РЕГУЛЯРНОГО ЗРОСТАННЯ

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Якщо для цілої функції f порядку $\rho \in (0;+\infty)$ з індикатором h для деякого $\rho_1 \in (0;\rho)$ існує множина $U \subset \mathbb{C}$, яка міститься в об'єднанні таких попарно неперетинних кругів $U_k = \{z : |z-a_k| < \tau_k\}$, $k \in \mathbb{N}$, що $\sum_{k\in\mathbb{N}} \tau_k < +\infty$, $\sum_{k\in\mathbb{N}} \tau_k |\log \tau_k| < +\infty$, і $\log|f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_1})$, $U \not\ni z = re^{i\varphi} \to \infty$, то для деякого $\rho_2 \in (0;\rho)$ рівномірно за $\varphi \in [0;2\pi]$ виконується $\int_1^r t^{-1} \log|f(te^{i\varphi})|\,dt = \frac{r^\rho}{\rho} h(\varphi) + o(r^{\rho_2})$, $r \to +\infty$.

Ключові слова: ціла функція покращеного регулярного зростання, виняткова множина, асимптотика.

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