Volume 6, Number 2, 2020 110

Construction of Open-Loop Electromechanical System Fundamental Matrix and Its Application for Calculation of State Variables Transients

Orest Lozynskyy, Yuriy Biletskyi, Andriy Lozynskyy, Volodymyr Moroz, Lidiya Kasha*

Lviv Polytechnic National University, 12, S. Bandery Str., Lviv, 79013, Ukraine

Received: August 07, 2020. Revised: September 11, 2020. Accepted: September 18, 2020.

Abstract

The article considers the methods of calculating the transition matrix of a dynamic system, which is based on the transient matrix representation by the matrix exponent and on the use of the system signal graph. The advantages of the transition matrix calculating using a signal graph are shown. The application of these methods to find the transition matrix demonstrated on the simple electromechanical system example. It is shown that the expression for the transition matrix as a matrix exponent completely corresponds to the expression found by means of the inverse matrix and based on the use of the signal graph. The transient matrix of a dynamical system thus found as a matrix exponent can be used to analyze processes in a system that is described by differential equations with integer derivatives. The formation of a transient matrix for the analysis of system processes, which is described by equations with fractional derivatives, is also considered. It is shown that the description of processes in systems with fractional derivatives based on the transient matrix and the representation of the fractional derivative in the form of Caputo-Fabrizio makes it possible to study coordinate transients without approximations in the description of the fractional derivative.

Keywords: control theory; electromechanical system; fractional derivative; linear system; state representation; transient matrix.

1. Introduction

The modern approach of the dynamic systems' simulation and researching is based on their time domain representation [1]. Description of the physical systems that are described by differential equations of the *n*-order traditionally move to a system of the *n* first order differential equations. Having written these equations in a compact vector and matrix form, we obtain a model in state variables [2], [3]. Note that the description of systems in the time domain is the basis of modern optimization methods for such systems [4].

The main characteristic for the time processes' calculation of the state and output variables is the transient matrix of a dynamic system [5]–[7]. The most widely used calculation methods of the dynamic system's transient matrix are based on:

- the method of the inverse matrix;
- the representation of the transient matrix by the matrix exponent [3], [8];
- the method based on the a signal graph of the system.

Each of these methods has some advantages and disadvantages. In particular, the use of inverse matrix system for higher order is quite a tedious task. When finding $L^{-1}\mathbf{\Phi}(t)$ as the matrix exponent, the accuracy of the obtained solution when using an expanding in a series $\mathbf{\Phi}(t) = \exp[At] = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$ significantly depends on the number of

* Corresponding author. Email address: lidiia.v.kasha@lpnu.ua

This paper should be cited as: O. Lozynskyy, Y. Biletskyi, A. Lozynskyy, V. Moroz, L. Kasha. Construction of open-loop electromechanical system fundamental matrix and its application for calculation of state variables transients. Energy Engineering and Control Systems, 2020, Vol. 6, No. 2, pp. 110–119. https://doi.org/10.23939/jeecs2020.02.110

members of the series [8]. The advantages of calculating a state transition matrix based on the signal graph become obvious if we consider the matrix differential equation of state $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU}$ and his Laplace transformation:

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{X}(0) + [s\mathbf{I} - \mathbf{A}]^{-1} \cdot \mathbf{B} \cdot \mathbf{U}(s). \quad (1)$$

Then, for $\mathbf{U}(s) = 0$, get $\mathbf{X}(s) = \mathbf{\Phi}(s) \cdot \mathbf{X}(0)$, and the state transition matrix will be the inverse Laplace transform from $\mathbf{\Phi}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}$. Transformation $\mathbf{\Phi}(s)$ in this case, we find using the signal graph after setting relationship between the state variable transform $X_i(s)$ and initial conditions $[x_1(0), x_2(0), \dots x_n(0)]$. The dependence of an arbitrary state variable $x_i(t)$ on the initial conditions is determined using the known Mason's formula. So, for an n^th^ order system we can write:

$$\begin{vmatrix} x_1(s) \\ x_2(s) \\ \vdots \\ x_n(s) \end{vmatrix} = \begin{vmatrix} \varphi_{11}(s) & \varphi_{12}(s) & \dots & \varphi_{1n}(s) \\ \varphi_{21}(s) & \varphi_{22}(s) & \dots & \varphi_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{n1}(s) & \varphi_{n2}(s) & \dots & \varphi_{nn}(s) \end{vmatrix} \cdot \begin{vmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{vmatrix}, \quad (2)$$

where $\varphi_{ij}(s) = \frac{x_i(s)}{x_j(0)} = \frac{s^{-1}\ P_{ij}\,\Delta_{ij}}{\Delta}$ are elements of the state transition matrix getting by the setting from the dependence; P_{ij} is path transfer coefficient (gain) from initial value $x_j(0)$ to variable x_i; Δ_{ij} is cofactor for the path P_{ij} ; Δ is determinant of the graph.

2. Presentation and discussion of the research work results

We demonstrate the application of the last two methods of finding the dynamic system's transition matrix for the electromechanical system (example from [6], [9]) shown in Fig. 1. State variables for this case are $y(s) = x_1 = \omega_M$ – angular motor speed; $x_2 = i_f$ – field current; x_3 – internal coordinate.

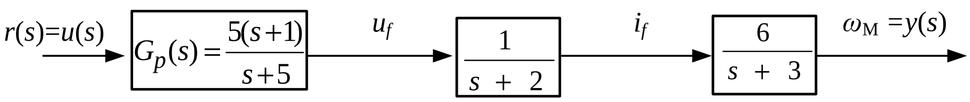

Fig. 1. Structure model of the electromechanical system [6], [9].

The signal graph of such a system with the state variables' initial values presented in Fig. 2.

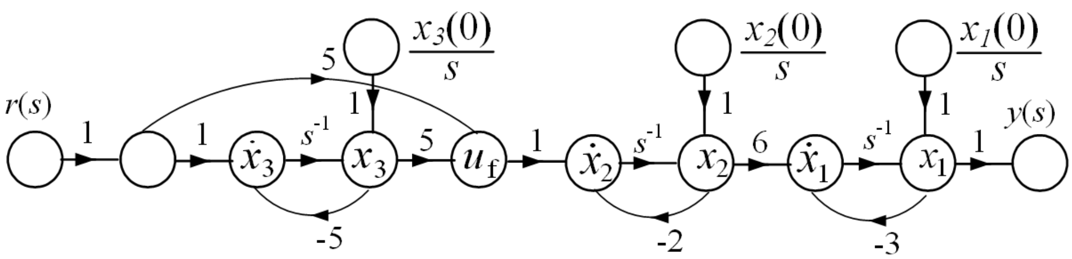

Fig. 2. The signal graph of a system with the state variables' initial values.

Given the regulator's structure of such an electromechanical system in the form $G_p(s) = \frac{5+5s^{-1}}{1+5s^{-1}}$, we find the relationship between the variables x_3 i $\dot{x}_2$ by removing the regulator's direct path from the transfer function (branch with factor 5) from the system input to the signal u_f. We can write the transfer function of such a system according to Mason's rule as equation

$$G(s) = \frac{5s^{-1}6s^{-2} + 5 \cdot 6s^{-2}}{1 - (-5s^{-1} - 2s^{-1} - 3s^{-1}) + 5s^{-1}2s^{-1} + 5s^{-1}3s^{-1} + 2s^{-1}3s^{-1} + 5s^{-1}2s^{-1}3s^{-1}}\ . \quad (3)$$

Then, when we must find the determinant of the graph and corresponds u_f . After these transformations, expression for $\mathbf{\Phi}(s)$ looks like:

$$\mathbf{\Phi}(s) = \begin{vmatrix} \frac{1}{(s+3)} & \frac{6}{(s+2)} - \frac{6}{(s+3)} & \frac{-40}{(s+2)} + \frac{60}{(s+3)} - \frac{20}{(s+5)} \\ 0 & \frac{1}{(s+2)} & \frac{-\frac{20}{3}}{(s+2)} + \frac{\frac{20}{3}}{(s+5)} \\ 0 & 0 & \frac{1}{(s+5)} \end{vmatrix} \quad (4)$$

and then the matrix $\mathbf{\Phi}(t)$ will look like:

$$\mathbf{\Phi}(t) = \begin{vmatrix} e^{-3t} & 6e^{-2t} - 6e^{-3t} & -40e^{-2t} + 60e^{-3t} - 20e^{-5t} \\ 0 & e^{-2t} & \frac{-20}{3}e^{-2t} + \frac{20}{3}e^{-5t} \\ 0 & 0 & e^{-5t} \end{vmatrix}. \quad (5)$$

It is easy to show that the transient function $\mathbf{\Phi}(t)$, which is found by the inverse Laplace transform of $\mathbf{\Phi}(s)$ in the form of an inverse matrix will look the same.

$$\mathbf{\Phi}(t) = L^{-1}[\mathbf{\Phi}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}]. \quad (6)$$

Now we find the transient matrix as a matrix exponent for this electromechanical system. When finding the transient matrix exponent, we will proceed from the following requirements:

- **First**, a system with zero control signal is called as open;
- **Second**, transition matrix that describes the transition of the system since the time t at time t_0 is an inverse matrix of the transition matrix, which characterizes the system's transition from time t_0 to time t.

The latter provision is easily proved by such statements:

$$\mathbf{X}(t) = \mathbf{\Phi}(t, t_0)\, x(t_0)\,. \quad (7)$$

Transient matrix $\mathbf{\Phi}(t, t_0)$ moves here the free trajectory of the system from the initial point t_0 in the point $\mathbf{X}(t)$, that correspond to time $\mathbf{X}(t)$. So, $\mathbf{X}(t_1) = \mathbf{\Phi}(t_1, t_0)\, \mathbf{X}(t_0)$ and $\mathbf{X}(t_2) = \mathbf{\Phi}(t_2, t_1)\, \mathbf{X}(t_1)$. Clearly, $\mathbf{\Phi}(t_0, t_0) = \mathbf{I}$, then $\mathbf{\Phi}(t_0, t_0) = \mathbf{\Phi}(t_0, t)\mathbf{\Phi}(t, t_0)^{-1} = \mathbf{I}$.

Thus, (7) can be written as

$$\mathbf{X}(t) = \mathbf{\Phi}(t, t_0)\, \mathbf{X}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t, t_1)\mathbf{B}\mathbf{U}(t_1)dt_1. \quad (8)$$

If we have only one input signal $\mathbf{U}(t)$, this equation can be written in the following expanded form [5]

$$x_i(t) = \sum_{j=1}^{n} \mathbf{\Phi}_{i,j}(t, t_0)\, x_j(t_0) + \int_{t_0}^{t} \sum_{j=1}^{n} \mathbf{\Phi}_{i,j}(t, t_1) b_j \mathbf{U}(t_1) dt_1\ , \quad (9)$$

where $\mathbf{\Phi}_{i,j}$ is element of matrix $\mathbf{\Phi}$ in i-th raw and j-th column; b_j is vector's component $\mathbf{B}$.

When all the initial values are zeros ($x_j(t_0) = 0$ for all j) and a single pulse is applied to the system $\mu_0(t)$ at time τ, there is easy to show that the close relationship is between the transition matrix and the impulse response:

$$x_i(t) = \int_{t_0}^{t} \sum_{j=1}^{n} \mathbf{\Phi}_{i,j}(t, t_1) b_j \mu_0(t_1) dt_1 = \sum_{j=1}^{n} \mathbf{\Phi}_{i,j}(t, t_1) b_j \text{ when } > t_0\ .$$

If we represent now the transition matrix of the state by the matrix exponent and take into account that free movement of linear system $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU}$, which according to [5] is described by the expression: $\mathbf{X}(t) = \mathbf{\Phi}(t, t_0)\, x(t_0)$, we will obtain the expression for the free component of the state variables vector $\mathbf{X}(t)$.

In the case of a stationary system, the general solution of the equation (10):

$$\mathbf{X}(t) = e^{\boldsymbol{A}(t-t_0)}\, X(t_0) + \int_{t_0}^{t} e^{\boldsymbol{A}(t-t_1)}\mathbf{BU}(t_1)dt_1\,. \tag{10}$$

If the matrix **A** is diagonal, the elements of which are eigenvalues or poles of the transfer function (solutions of the characteristic equation $det(\mathbf{\Lambda I}-\mathbf{A})=0$), then the fundamental matrix will have the form (11)

$$e^{\Lambda(t-t_0)} = \begin{vmatrix} e^{\lambda_1(t-t_0)} & 0 & \cdots & 0 \\ 0 & e^{\lambda_1(t-t_0)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n(t-t_0)} \end{vmatrix}. \tag{11}$$

When the matrix **A** isn't diagonal, it's need to find the matrix **T**, which converts the matrix into a diagonal matrix **Λ.** [7] shows that if all eigenvalues λ_i of matrix **Λ** are different, then such a matrix is transformed into a diagonal form by finding a nondegenerate matrix **T** such that $\mathbf{T}^{-1}\mathbf{AT}=\mathbf{\Lambda}$, $e^{\boldsymbol{A}(t-t_0)} = \boldsymbol{T}e^{\mathbf{\Lambda}(t-t_0)}\mathbf{T}^{-1}$ or $e^{\boldsymbol{A}t} = \boldsymbol{T}e^{\mathbf{\Lambda}t}\mathbf{T}^{-1}$.

To find the matrix **T** we use the condition under which it is necessary to find the vector **X** and the scalar value λ so that the equation $\mathbf{AX}=\mathbf{\Lambda X}$ is satisfied for any given quadratic matrix **A**. The values of those scalars for which this equation is satisfied are called the eigenvalues of the quadratic matrix **A**. We write the mentioned equation in the form $(\mathbf{A}-\lambda\mathbf{I})\mathbf{X}=0$ or $det(\mathbf{A}-\lambda\mathbf{I})=0$, which is a necessary and sufficient condition for the existence of non-trivial solutions of the equation. Note that this equation will hold when such a vector **V** is found in the state space, which is transformed by the matrix **A** to the nearest factor itself. So, we find the roots of the equation $det(\mathbf{A}-\lambda\mathbf{I})=0$, where for each value λ_i corresponds the vector V_i – eigenvector and $\mathbf{A}V_i=\lambda_i V_i$.

For the above-mentioned electromechanical system we can write vector-matrix

$$\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{vmatrix} = \mathrm{AX}+\mathrm{B}U = \begin{vmatrix} -3 & 6 & 0 \\ 0 & -2 & -20 \\ 0 & 0 & -5 \end{vmatrix} \cdot \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} + \begin{vmatrix} 0 \\ 5 \\ 1 \end{vmatrix} r(t)\,. \tag{12}$$

Now find $det(\mathbf{A}-\lambda\mathbf{I}) = \lambda^3+10\lambda^2+31\lambda+30=0$ has such roots $\lambda_1=-2;\ \lambda_2=-3;\ \lambda_3=-5.$

Similarly, to above algorithm we write matrixes **T** and $\mathbf{T}^{-1}$ in form:

$$\mathbf{T} = \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{6} & 0 & \frac{-1}{3} \\ 0 & 0 & \frac{-1}{20} \end{vmatrix} \text{ and } \mathbf{T}^{-1} = \begin{vmatrix} 0 & 6 & -40 \\ 1 & -6 & 60 \\ 0 & 0 & -20 \end{vmatrix}, \text{ considering that } det(\mathbf{T}) = \frac{1}{120}.$$

Thus, the transient matrix $\mathbf{\Phi}(t)$ using matrix exponent can be written as

$$\mathbf{\Phi}(t) = \begin{vmatrix} e^{-3t} & 6e^{-2t}-6e^{-3t} & -40e^{-2t}+60e^{-3t}-20e^{-5t} \\ 0 & e^{-2t} & \frac{-20}{3}e^{-2t}+\frac{20}{3}e^{-5t} \\ 0 & 0 & e^{-5t} \end{vmatrix}. \tag{13}$$

The expression (13) fully corresponds to expression $\mathbf{\Phi}(t)$, that was found by using the inverse matrix and based on the signal graph.

Calculation of the transition matrix element using signal graph requires constructing a model system based on Laplace transform, then search for Mason paths from the initial value of the variable $x_j(0)$ to variable x_i and following inverse Laplace transform to find $\mathbf{\Phi}(t)$. In the case of calculating $\mathbf{\Phi}(t)$ by the method of matrix exponent, Laplace transforms can be avoided, which can be considered as an advantage of such a method. This advantage is not pull down, but rather reinforced for the case when the characteristic equation of the system matrix will have complex-conjugate roots and, accordingly, the vectors will have complex components.

When finding the transient matrix $\mathbf{\Phi}(t)$ by calculating inverse matrix $\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1}$ or search for components $\mathbf{\Phi}(s)$ using signal graph followed by the inverse Laplace transform in both cases $\mathbf{\Phi}(t) = L^{-1}\mathbf{\Phi}(s)$, it is possible to analyze the frequency characteristics of the system too. Because matrix function of a complex variable *s* is a transfer function from control "*u*" to output "*y*" for the system

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU} \\ \mathbf{Y} = \mathbf{CX} \end{cases} \quad (14)$$

and is called the frequency response of the system $\mathbf{H}(j\omega) = \mathbf{C}(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$.

Using transient matrix $\mathbf{\Phi}(t)$, transients of changing state variables $\mathbf{X}(t)$ for open-loop (no control action $\mathbf{U} = -\mathbf{KX}$) electromechanical systems (Fig. 1) calculate by the following formula

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\,\mathbf{X}(0) + \int_0^t \Phi(t-\tau)\mathbf{D}_1 w(\tau)d\tau, \quad (15)$$

where $\boldsymbol{w}(t) = r(t)$ is vector of input signals (external perturbations or setting influences); $\mathbf{D}_1 = |0 \quad 5 \quad 1|^{\mathrm{T}}$ is matrix of these input signals.

So

$$\mathbf{X}(t) = \begin{vmatrix} e^{-3t} & 6e^{-2t} - 6e^{-3t} & -40e^{-2t} + 60e^{-3t} - 20e^{-5t} \\ 0 & e^{-2t} & \frac{-20}{3}e^{-2t} + \frac{20}{3}e^{-5t} \\ 0 & 0 & e^{-5t} \end{vmatrix} \cdot \begin{vmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{vmatrix} +$$

$$+\int_0^t \begin{vmatrix} e^{-3(t-\tau)} & 6e^{-2(t-\tau)} - 6e^{-3(t-\tau)} & -40e^{-2(t-\tau)} + 60e^{-3(t-\tau)} - 20e^{-5(t-\tau)} \\ 0 & e^{-2(t-\tau)} & \frac{-20}{3}e^{-2(t-\tau)} + \frac{20}{3}e^{-5(t-\tau)} \\ 0 & 0 & e^{-5(t-\tau)} \end{vmatrix} \times \begin{vmatrix} 0 \\ 5 \\ 1 \end{vmatrix} r(\tau)d\tau\,. \quad (16)$$

Multiplying the matrixes in (16), we get equations for state variables transient $x_1(t), x_2(t), x_3(t)$ of the studied electromechanical system. Graphs of such variables are given on Fig. 3 and Fig. 4.

$$x_1(t) = e^{-3t}x_1(0) + (6e^{-2t} - 6e^{-3t})\,x_2(0) + (-40e^{-2t} + 60e^{-3t} - 20e^{-5t}) \cdot$$
$$\cdot\, x_3(0) + \int_0^t \left(-10e^{-2(t-\tau)} + 30e^{-3(t-\tau)} - 20e^{-5(t-\tau)}\right) r(\tau)d\tau\,;$$

$$x_2(t) = e^{-2t}\,x_2(0) + \left(\frac{-20}{3}e^{-2t} + \frac{20}{3}e^{-5t}\right)\,x_3(0) + \int_0^t \left(\frac{-5}{3}e^{-2(t-\tau)} + \frac{20}{3}e^{-5(t-\tau)}\right) r(\tau)d\tau\,; \quad (17)$$

$$x_3(t) = e^{-5t}\,x_3(0) + \int_0^t \left(e^{-5(t-\tau)}\right) r(\tau)d\tau\,.$$

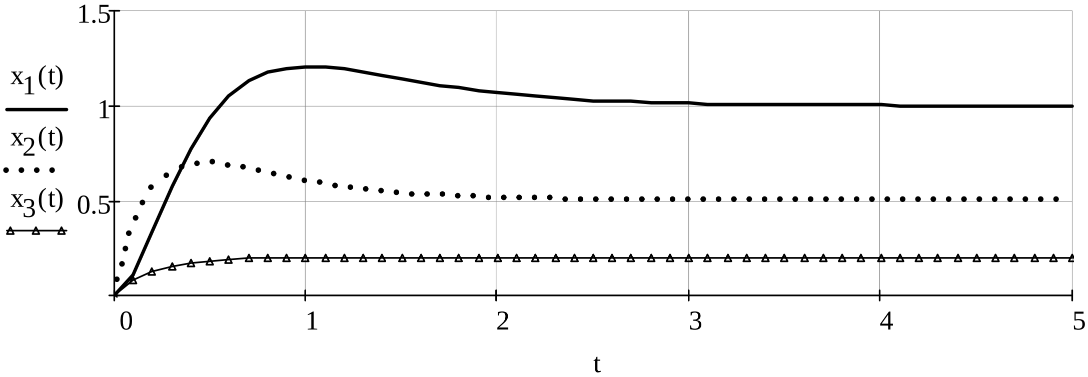

Fig. 3. Graphs of state variables transients at zero initial conditions $x_1(0) = x_2(0) = x_3(0) = 0$.

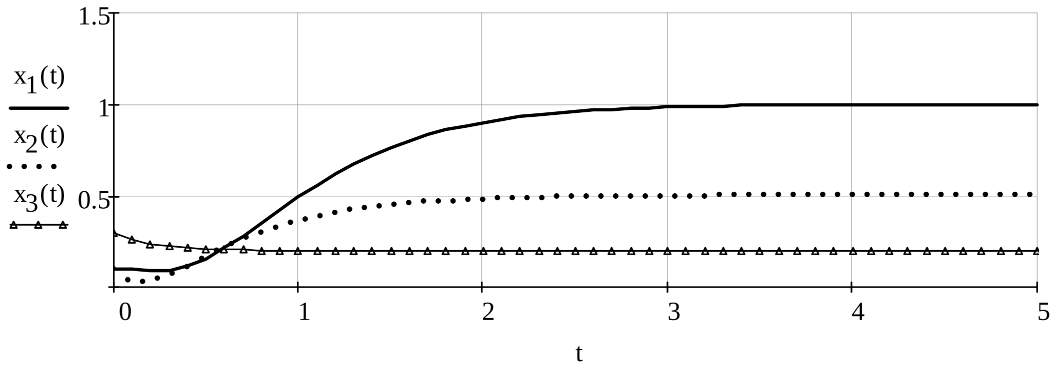

Fig. 4. Graphs of state variables transients at nonzero initial conditions $x_1(0) = 0.1$; $x_2(0) = 0.1$; $x_3(0) = 0.3$.

When $t = \infty$ the obtained value of the output signal (variable $x_1(t)$) corresponds to the step response of the system, which is described by the transfer function (18) and when $s = 0$, that is, for a steady process.

$$G(s) = \frac{30s + 30}{s^3 + 10s^2 + 31s + 30}\,. \tag{18}$$

Thus, found by this method, the transient matrix of a dynamic system in the matrix exponent form can be used to analyze processes in the system, which is described by a system of differential equations with integer derivatives.

Let us now consider the formation of a transient matrix for the transient analysis in a system described by equations with fractional derivatives.

There are several models for describing the fractional derivative today, but the most effective and suitable should be considered the model Caputo-Fabrizio, which is proposed in [10], according to which we can easy to show that for linear systems described by fractional derivatives, we can write

$$\frac{d^\alpha f(t)}{dt^\alpha} = \mathbf{A} \cdot \mathbf{X}(t)\,. \tag{19}$$

Now performing the Laplace transform [11] for both parts of (19) and taking into account the model of the description of the fractional derivative under the initial conditions $x(t_0) = x_0$, we get

$$\mathrm{L}\big(\mathbf{A} \cdot \mathbf{X}(t)\big) \leftrightarrow \frac{1}{1-\alpha}\left(\frac{1}{s+\beta}(\mathrm{s}\mathbf{X}(s) - \mathbf{X}_0)\right) = \mathbf{A} \cdot \mathbf{X}(s)\,, \tag{20}$$

where $\beta = \frac{\alpha}{1-\alpha}$.

After the transformations of equation (19), marking $\mathbf{N} = (\boldsymbol{I} - (1-\alpha)\boldsymbol{A})^{-1}$ and multiplying by $\mathbf{N}$ both parts of the equation (19) we obtain the expression to find $\mathbf{X}(s)$ and as a result of the inverse Laplace transform, the expression for $\mathbf{X}(t)$

$$\mathbf{X}(s) = (s\mathbf{I} - \alpha\mathbf{N}\mathbf{A})^{-1}\mathbf{N}\mathbf{X}_0\,, \tag{21}$$

where $e^{\alpha\mathbf{NA}t} = e^{\mathbf{A}t}$ is matrix exponent.

The algorithm for forming a transient matrix in the matrix exponent form through eigenvalues and eigenvectors of a matrix, which is an exponent, is given above in this article. The considered general principle of formation of the system's transition matrix, which is described by differential equations with fractional derivatives, allows performing similar researches, such in the case of the electromechanical system, which is described by integer derivatives. For example, to do this for a system which is described in a model with integer derivatives by a transfer function, we write its analogue when describing the process of fractional derivatives as (22):

$$W(s^\alpha) = \frac{b_3(s^\alpha)^3 + b_2(s^\alpha)^2 + b_1(s^\alpha)^1 + b_0(s^\alpha)^0}{(s^\alpha)^4 + a_3(s^\alpha)^3 + a_1(s^\alpha)^1 + a(s^\alpha)^0}. \tag{22}$$

Constructing the graph that corresponds to (22) we obtain a system of equations by state variables

$$\frac{d^\alpha x_1}{dt^\alpha} = x_2\ ; \quad \frac{d^\alpha x_2}{dt^\alpha} = x_3\ ; \quad \frac{d^\alpha x_3}{dt^\alpha} = x_4\ ;$$

$$\frac{d^\alpha x_4}{dt^\alpha} = r(t) - a_3x_4 - a_2x_3 - a_1x_2 - a_0x_1\ ;$$

$$y = b_0x_1 + b_1x_2 + b_2x_3 + b_3x_4$$

and the corresponding vector-matrix equation of the system in the form

$$\frac{d^\alpha}{dt^\alpha}\begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = \underbrace{\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{vmatrix}}_{\mathbf{A}} \cdot \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} + \underbrace{\begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}}_{\mathbf{B}} r(t)\ . \tag{23}$$

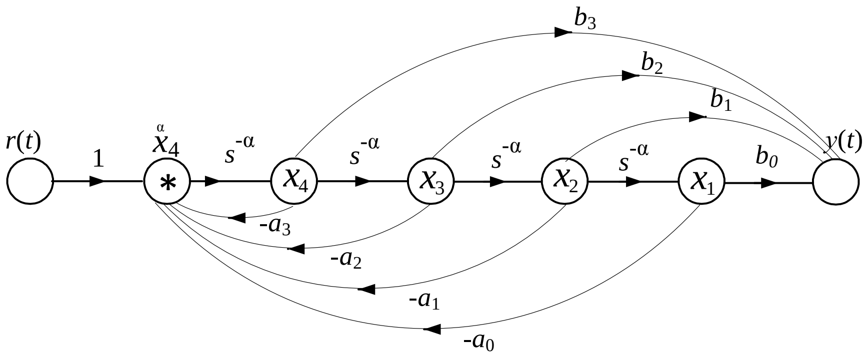

Fig. 5. Signal graph of the system represented by fractional derivatives.

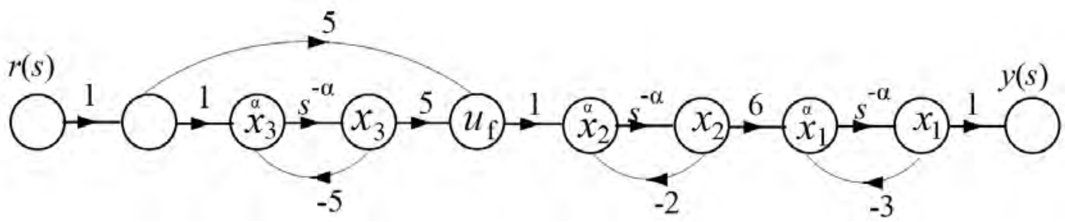

Fig. 6. Signal graph of the system represented by fractional derivatives.

For the electromechanical system (see Fig. 5), which is described by a system of equations with fractional derivatives with power $\alpha = 1/2$:

$$\frac{d^{\frac{1}{2}}x_1}{dt^{\frac{1}{2}}} = -3x_1 + 6x_2 + 0x_3\ ;$$

$$\frac{d^{\frac{1}{2}}x_2}{dt^{\frac{1}{2}}} = 0x_1 - 2x_2 - 20x_3 + 5r(t);$$

$$\frac{d^{\frac{1}{2}}x_3}{dt^{\frac{1}{2}}} = 0x_1 + 0x_2 - 5x_3 + 1r(t)\,.$$

We form a matrix **N** considering that $\mathbf{N} = (\mathbf{I} - (1-\alpha)\mathbf{A})^{-1}$. So taking into account that $\Delta = 35/2$, we will find matrix **N** and it component $\alpha\mathbf{NA}$

$$\mathbf{N} = \begin{vmatrix} \frac{2}{5} & \frac{3}{5} & -\frac{12}{7} \\ 0 & \frac{1}{2} & -\frac{10}{7} \\ 0 & 0 & \frac{2}{7} \end{vmatrix}^{\mathrm{T}};\ \alpha\mathbf{NA} = \begin{vmatrix} -\frac{3}{5} & \frac{6}{10} & -\frac{12}{7} \\ 0 & -\frac{1}{2} & -\frac{10}{7} \\ 0 & 0 & -\frac{5}{7} \end{vmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = \frac{-5}{7}$; $\lambda_2 = \frac{-3}{5}$; $\lambda_3 = \frac{-1}{2}$. Now let's write the expression for the transition matrix $\mathbf{\Phi}(t)$ through the diagonal matrix and the matrix $\mathbf{T}$ i $\mathbf{T}^{-1}$ as

$$\mathbf{\Phi}(t) = \begin{vmatrix} -20 & 1 & 6 \\ \frac{20}{3} & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} e^{-\frac{5}{7}t} & 0 & 0 \\ 0 & e^{-\frac{3}{5}t} & 0 \\ 0 & 0 & e^{-\frac{1}{2}t} \end{vmatrix} \cdot \begin{vmatrix} 0 & 0 & 1 \\ 1 & -6 & 60 \\ 0 & 1 & -\frac{20}{3} \end{vmatrix}, \tag{24}$$

where $\mathbf{T}$ i $\mathbf{T}^{-1}$ – found matrices through eigenvectors of the matrix $\alpha\mathbf{NA}$.

After the transformations (24), we get (25):

$$\mathbf{\Phi}(t) = \begin{vmatrix} e^{-\frac{3}{5}t} & -6e^{-\frac{3}{5}t} + 6e^{-\frac{1}{2}t} & -20e^{-\frac{5}{7}t} + 60e^{-\frac{3}{5}t} - 40e^{-\frac{1}{2}t} \\ 0 & e^{-\frac{1}{2}t} & \frac{20}{3}e^{-\frac{5}{7}t} - \frac{20}{3}e^{-\frac{1}{2}t} \\ 0 & 0 & e^{-\frac{5}{7}t} \end{vmatrix}, \tag{25}$$

Given expression (18) for $\mathbf{\Phi}^*(t)$ will look like

$$\mathbf{\Phi}^*(t) = \mathbf{\Phi}(t)\mathbf{N} = \begin{vmatrix} 0.4e^{-\frac{3}{5}t} & -2.4e^{-\frac{3}{5}t} + 3e^{-\frac{1}{2}t} & -\frac{40}{7}e^{-\frac{5}{7}t} + 24e^{-\frac{3}{5}t} - 20e^{-\frac{1}{2}t} \\ 0 & 0.5e^{-\frac{1}{2}t} & \frac{40}{21}e^{-\frac{5}{7}t} - \frac{10}{3}e^{-\frac{1}{2}t} \\ 0 & 0 & \frac{2}{7}e^{-\frac{5}{7}t} \end{vmatrix}. \tag{26}$$

It is not difficult to show that for a system described by a vector-matrix equation $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU}$, the expression for the transition matrix is written as

$$\mathbf{\Phi}^*(t)\cdot\alpha\cdot\mathbf{B} = \begin{vmatrix} -\frac{20}{7}e^{-\frac{5}{7}t} + 6e^{-\frac{3}{5}t} - 2.5e^{-\frac{1}{2}t} \\ \frac{20}{21}e^{-\frac{5}{7}t} - \frac{5}{12}e^{-\frac{1}{2}t} \\ \frac{1}{7}e^{-\frac{5}{7}t} \end{vmatrix}, \tag{27}$$

where $\mathbf{B} = |0 \quad 5 \quad 1|^{\mathrm{T}}$.

After integrating (27) using boundaries from 0 to t, we obtain the following dependence

$$x_1(t) = \int_0^t \left(-\frac{20}{7}e^{-\frac{5}{7}(t-\tau)} + 6e^{-\frac{3}{5}(t-\tau)} - 2.5e^{-\frac{1}{2}(t-\tau)}\right)d\tau = 1 - 4e^{-\frac{5}{7}t} - 10e^{-\frac{3}{5}t} + +5e^{-\frac{1}{2}t};$$

$$x_2(t) = \int_0^t \left(\frac{20}{21}e^{-\frac{5}{7}(t-\tau)} - \frac{5}{12}e^{-\frac{1}{2}(t-\tau)}\right)d\tau = \frac{1}{2} - \frac{4}{3}e^{-\frac{5}{7}t} + \frac{5}{6}e^{-\frac{1}{2}t};$$

$$x_3(t) = \int_0^t \left(\frac{1}{7}e^{-\frac{5}{7}(t-\tau)}\right)d\tau = \frac{1}{5} - \frac{1}{5}e^{-\frac{5}{7}t},$$

for the integral component of (15) and for the component due to the initial values of the vector $\mathbf{X}(0)$

$$\mathbf{\Phi}^*(t)\mathbf{X}(0)=\begin{vmatrix} 0.4e^{-\frac{3}{5}t} & 3e^{-\frac{1}{2}t}-2.4e^{-\frac{3}{5}t} & 24e^{-\frac{3}{5}t}-\frac{40}{7}e^{-\frac{5}{7}t}-20e^{-\frac{1}{2}t} \\ 0 & 0.5e^{-\frac{1}{2}t} & \frac{40}{21}e^{-\frac{5}{7}t}-\frac{10}{3}e^{-\frac{1}{2}t} \\ 0 & 0 & \frac{2}{7}e^{-\frac{5}{7}t} \end{vmatrix}\begin{vmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{vmatrix}.$$

Thus, based on the obtained expressions according to the above approach, we can write the equation to find the time dependences of the state variables $x_1(t), x_2(t), x_3(t)$ and build their plots (see Fig. 7):

$$x_1(t)=0.4e^{-\frac{3}{5}t}x_1(0)+\left(3e^{-\frac{1}{2}t}-2.4e^{-\frac{3}{5}t}\right)x_2(0)+$$
$$+\left(24e^{-\frac{3}{5}t}-\frac{40}{7}e^{-\frac{5}{7}t}-20e^{-\frac{1}{2}t}\right)x_3(0)+\left(1-4e^{-\frac{5}{7}t}-10e^{-\frac{3}{5}t}+5e^{-\frac{1}{2}t}\right);$$
$$x_2(t)=0.5e^{-\frac{1}{2}t}x_2(0)+\left(\frac{40}{21}e^{-\frac{5}{7}t}-\frac{10}{3}e^{-\frac{1}{2}t}\right)x_3(0)+\left(\frac{1}{2}-\frac{4}{3}e^{-\frac{5}{7}t}+\frac{5}{6}e^{-\frac{1}{2}t}\right);$$
$$x_3(t)=\frac{2}{7}e^{-\frac{5}{7}t}x_3(0)+\left(\frac{1}{5}-\frac{1}{5}e^{-\frac{5}{7}t}\right).$$

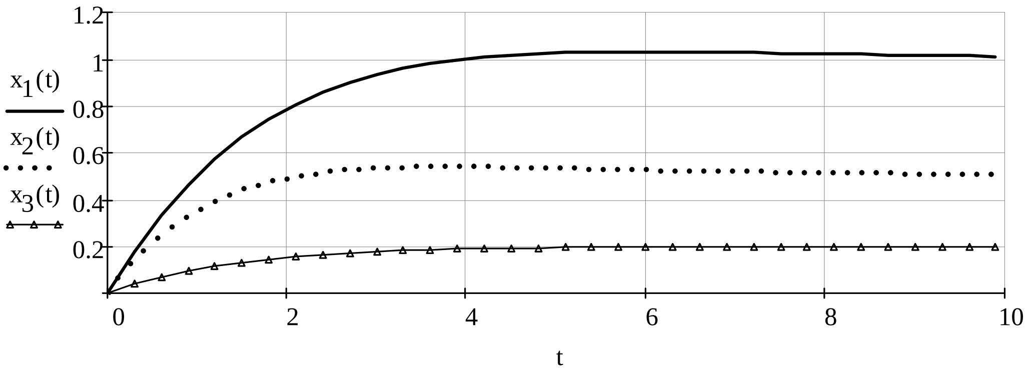

Fig. 7. Plot of the state variables transient at zero initial conditions $x_1(0)=x_2(0)=x_3(0)=0$.

Analyzing the built plots of state variables x_1(t); x_2(t) ; x_3(t) to describe the transients of the considered electromechanical system by its models described by differential equations with integer (Fig. 3 and Fig. 4) and fractional (Fig. 7) derivatives, we conclude that the static characteristics of these transients are nearly same for both models. As for the dynamic properties of these transients, for a system whose model is formed by equations with integer derivatives for state variables x_1(t) and x_2(t) under a given control there is some overshoot (up to 20 %), and for a system whose model contains fractional derivatives such overshoot is absent. But in the system described by the model in integer derivatives of the speed of transients $x_1=2.5\ s$; $x_2=2\ s$ and $x_3=1\ s$, and for the system described by the model in fractional derivatives, the speed of processes is less than about two times smaller, due to the roots obtained for the integer model $\lambda_1=-2$; $\lambda_2=-3$; $\lambda_3=-5$, and roots in the fractional description of the system $\lambda_1=\frac{-5}{7}$; $\lambda_2=\frac{-3}{5}$; $\lambda_3=\frac{-1}{2}$. As is known, the speed is determined by the magnitude of the real root or real part of the complex-conjugate root, the larger the value of this negative root, the greater the speed.

3. Conclusion

The calculation of the transients of linear stationary dynamical systems, as highlighted in the works of many authors, is carried out in two ways: using the transition state matrix and using a discrete approximation of the state equations. In our opinion, the first way is more promising for electromechanical systems.

The use of transition matrices to find the transients of state variables $x_1(t) \dots x_n(t)$ allows having analytical expressions of these processes in systems described by models with integer and fractional derivatives. This leads to their widespread use in problems with predictors, in problems with estimators and in problems with transport delay for formation of necessary properties of such systems dynamics.

The description of processes in systems with fractional derivatives based on the transient matrix and the representation of the fractional derivative in the form of Caputo-Fabrizio makes it possible to study coordinate transients without any approximations in the description of the fractional derivative, in particular, as done according to Oustaloup model.

References

[1] Kim D. Automatic Control Theory. Vol. 1. Linear Systems. M.: Phismatlit, 2003. 288 p. (in Russian)

[2] Methods of classic and modern automatic control theory: The textbook in 3 Vol. Vol. 1: Analysis and statistical dynamics of automatic control systems / Edit. N. D. Egupov. M.: Publishing House MSTU N. E. Bauman, 2000. 748 p., FL. (in Russian)

[3] Derek Rowell. Analysis and Design of Feedback Control Systems: Time-Domain Solution of LTI State Equations. October 2002. – http://web.mit.edu/2.14/www/Handouts/StateSpaceResponse.pdf

[4] Methods of classic and modern automatic control theory: The textbook in 3 Vol. Vol. 2: Synthesis of regulators and theory of optimization of automatic control systems / Edit. N. D. Egupov. M.: Publishing House MSTU N. E. Bauman, 2000. 736 p., FL. (in Russian)

[5] EE 580 – Linear Control Systems VI. State Transition Matrix. – Department of Electrical Engineering. Pennsylvania State University, 2010. https://www.coursehero.com/file/12966838/SUMMARY-of-State-Transition-Matrix-Fundamental-Matrix/

[6] Richard C. Dorf, Robert H. Bishop. Modern Control Systems. 12th Edition. – USA: Pearson, 2010. [ISBN-13: 978-0136024583]. 1104 p.

[7] Jay C. Hsu, Andrew U. Meyer. Modern Control Principles and Applications. –Published by McGraw Hill Book Co, NY, 1968. [ISBN 10: 0070306354 / ISBN 13: 9780070306356]. 769 p.

[8] Leon O. Chua, Pen-Min Lin. Computer-aided analysis of electronic circuits: algorithms and computational techniques. – Englewood Cliffs, N. J.: Prentice-Hall, 1975. 737 p. [ISBN 0131654152].

[9] Pulyer Y. M. Electromagnetic Devices for Motion Control, ©1992 Springer-Verlag, New York, Inc. [ISBN13: 978-1461277231]. 468 p.

[10] Michele Caputo, Mauro Fabrizio. A new Definition of Fractional Derivative without Singular Kernel. Progress in Fractional Differentiation and Applications. An International Journal, 2015. http://dx.doi.org/10.12785/pfda/010201

[11] Hong Li, Jun Cheng, Hou-Biao Li, Shou-Ming Zhong. Stability Analysis of a Fractional-Order Linear System Described by the Caputo-Fabrizio Derivative // Mathematics, 2019, 7, 200; doi:10.3390/math7020200. – www.mdpi.com/journal/mathematics.

Формування фундаментальної матриці відкритої електромеханічної системи і її застосування для розрахунку часових процесів змінних стану

Орест Лозинський, Юрій Білецький, Андрій Лозинський, Володимир Мороз, Лідія Каша

Національний університет “Львівська політехніка”, вул. С. Бандери, 12, Львів, 79013, Україна

Анотація

У статті розглянуто методи обчислення перехідної матриці динамічної системи, які ґрунтуються на представленні фундаментальної матриці матричною експонентою та на використанні сигнального графа системи. Показані переваги обчислення перехідної матриці стану на основі використання сигнального графа. Продемонстровано застосування цих методів для знаходження перехідної матриці на прикладі простої електромеханічної системи. Показано, що вираз для перехідної матриці як матричної експоненти повністю відповідає виразу, що знайдений за допомогою оберненої матриці та на основі використання сигнального графа. Знайдену таким чином фундаментальну матрицю динамічної системи як матричну експоненту можна використовувати для аналізу процесів у системі, яка описується диференціальними рівняннями з цілочисельними похідними. Також розглянуто формування фундаментальної матриці для аналізу процесів у системі, яка описується рівняннями з дробовими похідними. Показано, що опис процесів у системах із дробовими похідними на основі фундаментальної матриці та представлення дробової похідної у формі Caputo-Fabrizio дає можливість досліджувати перехідні процеси координат без наближень в описі дробової похідної.

Ключові слова: дробові похідні; електромеханічні системи; лінійні системи; опис у просторі станів; перехідна матриця стану; теорія автоматичного керування.