Exponentially Fitted Methods on Layer-Adapted Mesh for Singularly Perturbed Delay Differential Equations

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***Abstract* – The purpose of this study is to present a uniform finite difference method for numerical solution of a initial value problem for quasi-linear second order singularly perturbed delay differential equation. A numerical method is constructed for this problem which involves appropriate piecewise-uniform Shishkin mesh on each time subinterval. The method is shown to uniformly convergent with respect to the perturbation parameter. A numerical experiment illustrate in practice the result of convergence proved theoretically.**

Key words – The finite difference method, Appropriate piecewise-uniform Shishkin mesh.

I. Introduction

Consider an initial value problem for the linear second order singularly perturbed delay differential equation

$$\varepsilon u''(t)+a(t)u'(t)+f(t,u(t),u(t-r))=0,\ t\in I, \quad (1)$$

$$u(t)=\varphi(t),\ t\in I_0, \quad (2)$$

$$u'(0)=A/\varepsilon, \quad (3)$$

where $I=(0,T], I_0=(-r,0]$. $0<\varepsilon\le 1$ is the perturbation parameter, $a(t)\ge\alpha>0,\ f(t)$ and $\varphi(t)$ are given sufficiently smooth functions satisfying certain regularity conditions to be specified and r is a constant delay.

Delay differential equations play an important role in themathematical modelling of various practical phenomena in thebiosciences and control theory. Any system involving a feedback control will almost always involve time delays. These arise because a finite time is required to sense information and then react to it. A singularly perturbed delay differential equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay term [1-4]. Such problems arise frequently in the mathematical modelling of various practical phenomena, for example, in the modelling of several physical and biological phenomena like the optically bistable devices [5], description of the human pupil-light reflex [6], a variety of models for physiological processes or diseases and variational problems in control theory where they provide the best and in many cases the only realistic simulation of the observed phenomena[7].An overview of numerical treatment for first and second order singularly perturbed delay differential equations, may be obtained in [8-15](see,also references therein).

The numerical analysis of singular perturbation cases has always been far from trivial because of the boundary layer behavior of the solution. Such problems undergo rapid changes within very thin layers near the boundary or inside the problem domain. It is well known that standard numerical methods for solving singular perturbation problems do not give satisfactory result when the perturbation parameter is sufficiently small. Therefore, it is important to develop suitable numerical methods for these problems, whose accuracy does not depend on the perturbation parameter, i.e. methods that are uniformly convergent with respect to the perturbation parameter [16-21].

In a singularly perturbed delay differential equation, one encounters with two difficulties, one is because of occurrence of the delay term and another one is due to presence of perturbation parameter. To overcome the first difficulty, we employed the numerical method of steps [2] for the delay argument which converted the problem to a initial value problem for a singularly perturbed differential equation. Then we constructed a numerical scheme based on finite difference method on non uniform Shishkin mesh for the numerical solution.

In the present paper we discretize the problem(1)-(3) using a numerical method, which is composed of an exponentially fitted difference scheme on piecewise uniform Shishkin mesh on each time-subinterval. In section 2, we state some important properties of the exact solution. In section 3, we describe the finite difference discretization and introduce the piecewise uniform mesh. In section 4, we present convergence analysis for approximate solution. Uniform convergence is proved in the discrete maximum norm. Some numerical results are being presented in section 5. The technique to construct discrete problem and error analysis for approximate solution is similar to those in [8,9,22,23].

II. Discretization and Mesh

In this section, we construct a numerical scheme for solving the initial value problem (1)-(3). We propose the following difference scheme for approximation (1)-(3).

$$\varepsilon\theta_i\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+a_i\frac{y_{i+1}-y_{i-1}}{2h}, +f(t_i,y_i,y_{i-N})=0$$
$$i=1,2,...,N-1, \quad (4)$$

$$y_i=\varphi_i,\ -N\le i\le 0,\quad \varepsilon\sigma(y_1-y_0)-Ah=0, \quad (5)$$

where θ_i and σ are defined by

$$\theta_i=\frac{ha_i}{2\varepsilon}\coth\left(\frac{ha_i}{2\varepsilon}\right),\ \sigma=\frac{ha_0}{\varepsilon}/\left(1-e^{\left(\frac{ha_0}{\varepsilon}\right)}\right)$$

The difference scheme (1)-(3), in order to be $\varepsilon-$uniform convergent, we will use the Shishkin mesh. For the even number N , the piecewise uniform mesh $\omega_{N,p}$ divides each of the interval $[r_{p-1},\sigma_p]$ and $[\sigma_p,r_p]$ into N/2 equidistant subintervals, where the transition point σ_p, which separates the fine and coarse portions of the mesh is obtained by $\sigma_p=r_{p-1}+\min\{r/2,\alpha^{-1}\varepsilon\ln N\}$.

III. Convergence Analysis

We now estimate the approximate error $z_i = y_i - u_i$, which satisfies the discrete problem

$$\varepsilon\theta_i \frac{z_{i+1} - 2z_i + z_{i-1}}{h^2} + a_i \frac{z_{i+1} - z_{i-1}}{2h}$$

$$+ f(t_i, y_i, y_{i-N}) - f(t_i, u_i, u_{i-N}) = R_i,, \quad (6)$$

$$i = 1, 2, ..., N,$$

$$\begin{gathered} z_i = 0, -N \le i \le 0, \\ \varepsilon\sigma(z_1 - z_0) - Ah + r^0 = 0 \end{gathered}, \quad (7)$$

where R_i and r^0 are the truncation errors.

IV. Theorem

The continuously differentiable function f(t,u,v) satisfies the regularity

conditions and the derivative $f(t,u,v)$ is bounded for $\frac{\partial}{\partial t} f(t,u,v)$ is bounded for given interval. Then the following estimate holds

$$|y_i - u_i| \le CN^{-1} \ln N, \; 0 \le i \le N.$$

V. Numerical Results

In this section, a simple numerical example is devised to verify the validity for the proposed method. Consider the test problem wit

$$a = 1, f = -u(t-1) + t^2, T = 2, \varphi = 1 + t, A = -1.$$

We use the double mesh principle to estimate the errors andcompute the experimental rates of convergence in our computed solution, i.e. We compare the computed solution with the solution on a mesh that is twice as fine [17].

Maximum errors and rates of convergence on $\omega_{N,1}$ and $\omega_{N,2}$

ε	N=64	N=128	N=256	N=512	N=1024
2^{-1}	0,0079814	0,0040103	0,0020101	0,0010063	0,0005033
2^{-8}	0,0191427	0,0114094	0,0063031	0,0037425	0,0020882
2^{-16}	0,0190685	0,0113652	0,0065754	0,0037280	0,0020801

Conclusion

In this study we have presented a numerical approach to solve a semi-linear singularly perturbed first-order delay differential equation, using an exponentially fitting factor for the difference scheme. We proposed an exponentially fitted difference scheme on piecewise-uniform Shishkin mesh on each time subinterval. We have shown that the method displays uniform convergence with respect to the perturbation parameter for numerical approximation of the solution.

References

[1] R. Bellman, K.L. Cooke, Differential-Difference Equations, Academy Press, New York, 1963.

[2] R.D. Driver, Ordinary and Delay Differential Equations, Belin-Heidelberg, New York, Springer, 1977.

[3] A. Bellen, M. Zennaro, Numerical methods for delay differential equations, Oxford University Press, Oxford, 2003.

[4] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, New York, 1993.

[5] S.N.Chow, J.Mallet-Paret, Singularly perturbed delay-differential equations, in: J.Chandra, A.C.Scott (Eds.), Coupled Nonlinear Oscillators, North-Holland, Amsterdam, 1983, pp.7-12.

[6] A. Longtin, J. Milton, Complex oscillations in the human pupil light reflex with mixed and delayed feedback, Math. Biosci. 90 (1988)183-199.

[7] M.C. Mackey, L. Glass, Oscillation and chaos in physiological control systems, Science, 197(1977)287-289.

[8] G.M. Amiraliyev, F. Erdogan, Uniform numerical method for singularly perturbed delay differential equations, J.Comput. Math. Appl. 53(2007)1251-1259.

[9] G.M. Amiraliyev, F. Erdogan, Difference schemes for a class of singularly perturbed initial value problems for delay differential equations, Numer. Algorithms, 52, 4(2009) 663-675.

[10] I.G. Amiraliyeva, Fevzi Erdogan, G.M.Amiraliyev, A uniform numerical method for dealing with a singularly perturbed delay initial value problem, Applied Mathematics Letters, 23,10(2010)1221-1225 .

[11] S. Maset, Numerical solution of retarded functional differential equations as abstract Cauchy problems, J. Comput. Appl. Math. 161(2003)259-282.

[12] B.J. MacCartin, Exponential fitting of the delayed recruitment/renewal equation. J. Comput. Appl. Math., 136(2001)343-356.

[13] M.K. Kadalbajoo, K.K. Sharma, $\varepsilon-$uniform fitted mesh method for singularly perturbed differential difference equations with mixed type of shifts with layer behavior, Int. J. Comput. Math. 81 (2004)49-62.

[14] C.G. Lange, R.M. Miura, Singular perturbation analysis of boundary-value problems for differential difference equations, SIAM J. Appl. Math. 42(1982)502-531.

[15] H. Tian, The exponential asymptotic stability of singularly perturbed delay differential equations with a bounded lag, J. Math. Anal. Appl. 270(2002)143-149.

[16] E. R. Doolan, J.J.H. Miller, and W. H. A. Schilders, Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole, Press, Dublin, (1980).

[17] P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E.O'Riordan and G.I.Shishkin, Robust Computational Techniques for Boundary Layers, Chapman-Hall/CRC, New York, (2000).

[18] H.G. Roos, M. Stynes and L. Tobiska, Numerical Methods for Singularly Perturbed Differential Equations, Convection-Diffusion and Flow Problems, Springer Verlag, Berlin, (1996).

[19] J.J.H. Miller, E. O'Riordan, G.I. Shishkin, Fitted Numerical Methods for Singular Perturbation Problems. Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions, World Scientific, Singapore, 1996.

[20] T. Linss, M. Stynes, A hybrid difference scheme on a Shishkin mesh for linear convection-diffusion problems, Appl. Numer. Math 31 (1999) 255-270.