UKRAINIAN JOURNAL OF MECHANICAL ENGINEERING AND MATERIALS SCIENCE
Vol. 8, No. 1, 2022

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MATHEMATICAL MODEL OF DYNAMICS OF VIBRATING SYSTEMS WORKING ENVIRONMENTS

Received: January 28, 2022 / Revised: March 16, 2022 / Accepted: March 30, 2022

https://doi.org/10.23939/ujmems2022.01.044

Abstract. Using the apparatus of the special periodic Ateb-functions in combination with the asymptotic methods of nonlinear mechanics, the nonlinear mathematical models of motion of working environment of the oscillation system, which dependences take into account resilient and viscid making tensions from descriptions of the deformation state of environment, her physical and mechanical properties and features of co-operation of environment with the oscillation system, are worked out. The nonlinear model for describing the dynamics of the working environment of oscillating systems is more flexible, because the nonlinearity index, which depends on the type of working load, significantly affects the results of the oscillating loading process. It allows us to take into account the type of load, and, accordingly, increase the level of adequacy of the constructed analytical model of the oscillatory process that needs to be investigated. Taking into account this model, the study of various processes in oscillating systems can be carried out, in particular in different modes of vibration processing.

Keywords: oscillation, asymptotic methods, environment, mathematical model, nonlinear mechanics

Introduction and Problem Statement

Processing of raw materials and semi-finished products is common in various production cycles. Intermediate stages of processing of bulk raw materials include separation, mixing, transportation, dosing, compaction, etc. The use of vibrating equipment in these processes and the direct impact of vibration on the processed raw materials, contribute to improving the quality of raw materials. In particular, due to constant shaking, the particles of raw materials are loosened and a high degree of their separation by physical and mechanical properties is achieved. The efficiency of the passage of bulk raw material's particles through the holes of the sieve is influenced by geometric dimensions, mass, structure of particles, their amplitude and frequency of oscillation, the interaction of particles with each other. Accordingly, the productivity of the separation process is affected not only by the design parameters of the vibrating separator, but also listed above characteristics of the loaded raw materials.

To date, a large number of studies and simulations of the movement of the bulk medium on the sieve of the vibrating separator. It was considered on the basis of many hypotheses, in particular, as the motion of a single particle, the motion of a set of particles, the motion of a solid body, the layering of layers of particles, etc. However, given that bulk raw materials perform complex spatial motion on the vibrating separator sieve, none of these hypotheses describes the process of the vibration separation in full. Therefore, the implementation of further research is still relevant.

Review of Modern Information Sources on the Subject of the Paper

To date, a number of models of the working environment of vibration systems have been developed, which to some extent reflect the physics of the studied vibration process. Work on the calculation of the dynamics of vibration systems can be divided into two main types: 1) work in which the environment is taken into account by the attached mass; 2) work in which the interaction of the environment and the vibration system is determined by other considerations. In these works there are those in which the problems of the dynamics of the working environment are considered in one-dimensional, flat and spatial formulation. In conducting many theoretical studies of the motion of the working environment of the vibration system, their authors tried to simplify, linearize the mathematical apparatus in solving these dynamic problems. These representations, on the one hand, made it possible to obtain simple relationships between some parameters of the medium, and on the other hand, did not reflect such properties of the medium as elasticity, viscosity, density change, and its motion. There are many ways to approach the basic problems of the dynamics of the bulk environment of vibrating systems by modelling it, but existing models are quite limited to use their assumptions [1].

Main Material Presentation

Construction of a nonlinear mathematical model of the working environment of vibration systems that implement vibration technologies of separation, grinding, mixing, compaction, transportation, surface treatment of products, vibration control technologies on systems and mechanisms for further study to improve the efficiency of vibration machines, devices and mechanisms and relevant technological processes. In recent years, the hypothesis of material's motion as a continuous medium, the material of which satisfies Voigt's linear law, has been used to study the motion of the working environment of vibration systems [1, 2]. This choice of model is primarily due to the fact that the dynamic processes in the medium are described by linear differential equations with partial derivatives, the study method of which is well developed. The latter in many cases prompted, in research, to linearize the corresponding nonlinear elastic characteristics of linear or completely neglect them. However, such “linearization” in many cases does not always reflect the real picture of the processes of a dynamic system. Representing the mixture of parts to be machined with the workpieces as a homogeneous continuous medium will be fair with sufficient accuracy for the small size of the first. In addition, assuming that the parts to be machined and the workpieces made of the same material (for example, when vibrating or deburring metal parts to be machined interact with steel balls), this homogeneous mixture can be given such properties of a solid material such as elasticity and viscosity, using Voigt's linear law. Stresses and strains that occur in it in the case of uniaxial stress are related:

$$\sigma = E \cdot \zeta + k_0 \left(\frac{d\zeta}{dt} \right), \tag{1}$$

where σ – normal voltage in the Vocht model; ζ – relative deformation of the model along the axis ξ; E – modulus of elasticity of the model material; k_0 – constant, which characterizes the viscosity properties of the model (the coefficient of viscosity of the material).

Using Voigt's generalized law, which can be presented in nonlinear form by introducing a power exponent of nonlinearity (it will depend on the physical and mechanical properties of the environment) in the corresponding terms, the mathematical apparatus of nonlinear differential equations with partial derivatives and the method of separation of Fourier variables, a model of motion of the working environment of the vibration system is built. The use of Voigt's law in nonlinear formulation made it possible to build on its basis nonlinear equations, and consequently their solutions of the medium motion, which are most adequate to reflect the physics of its motion during the vibration system compared to linear equations on the one hand and simple in its form – on the other.

Hypotheses for building a model of the working environment of the vibration system:

1. The material of the medium is continuous and homogeneous, presented as a layer of flat elastic-plastic beams, the thickness of which is much less than the length and which are in contact with the

working body of the vibrating system: a) as hinged beams, b) in the form of rigid contact, c) in the form of elastic contact. Such types of contact make it possible to take into account in the mathematical model various forms of interaction of the environment with the working body of the vibrating system.

2. The medium moves in layers and is in complex motion (in the plane of motion of the container): the portable motion of the medium layer is the movement together with the container of the vibrating system, the relative motion of the medium layer is oscillations relative to the horizontal axis of the layer ξ.

3. The material of the medium satisfies Voigt's nonlinear law. Two cases were considered: a) nonlinearity of the viscous stress component of Voigt's law (2), b) nonlinearity of the elastic stress component of Voigt's law (3), by introducing the power index of nonlinearity ν in the corresponding terms:

$$\sigma = E\zeta + k_0\left(\frac{d\zeta}{dt}\right)^{\nu+1}, \tag{2}$$

$$\sigma = E(\zeta)^{\nu+1} + k_0\frac{d\zeta}{dt}, \tag{3}$$

where σ – normal stress in the medium layer; $\zeta = \frac{\partial u}{\partial \xi}$ – deformation of the medium layer relative to the longitudinal axis of the layer ξ; $u = u(\xi, t)$ – moving along the axis ξ of arbitrary cross section of the model of the environment for some time t; E, k_0, ν – constants, characterizing the viscous and elastic properties of the medium.

4. The force of internal friction R in the layer of the medium, and also between its individual layers is determined by Bolotin's law [3]:

$$R = u_t\left(B + B_0 u^2\right), \tag{4}$$

where B, B_0 – constants, which are determined by the type of the environment; $u_t = \frac{\partial u}{\partial t}$

Then the equation of longitudinal nonlinear oscillations – the equation of relative motion of the middle layer's cross section of the vibration system will look like [1, 3]:

a) for the stress in the layer of the form (2)

$$u_{tt} - \alpha^2 u_{\xi\xi} - \beta\left(u_{\xi\xi t}\right)^{\nu+1} = f(t) + u_t\left(\vartheta + \delta u^2\right); \tag{5}$$

b) for the stress in the layer of the form (3) –

$$u_{tt} - \alpha^2\left(u_{\xi\xi}\right)^{\nu+1} - \beta u_{\xi\xi t} = f(t) + u_t\left(\vartheta + \delta u^2\right), \tag{6}$$

where $\alpha^2 = \frac{E}{\rho}$; $\beta = \frac{k_0}{\rho}$; $\vartheta = \frac{B_0}{\rho \cdot F}$; $\delta = \frac{B}{\rho \cdot F}$; F – the cross-sectional area of the layer of the model environment; ρ – equivalent density of the environment model; $f(t)$ – inertial load, which according to [4] is accepted harmonic type with amplitude b_1, (fluctuation amplitude of vibration system) $\mu = \frac{\pi n}{30}$, where n – the number of revolutions per minute of the motor drive vibration system), ie $f(t) = b_1 \sin \mu t$.

When considering equations (5) and (6), it is assumed that the forces of viscous friction are small compared to the nonlinear-elastic (reducing) force in the same layer, ie $\beta, \vartheta, \delta << \alpha^2$.

Separate studies of the dynamics of the working environment based on equations (5) and (6) have been conducted. Equation (5) also investigates longitudinal nonlinear oscillations of the medium layer during its modeling by a homogeneous elastic-plastic beam, the length of which far exceeds the thickness of the hinged and elastic fixing of the ends (corresponding contact of the medium with the container).

The influence of parameters characterizing nonlinear elastic properties of the medium on the dynamic characteristics of the medium layer is studied in order to bring the model closer to the reflection of real vibrational processes occurring in the medium and to compare the corresponding characteristics of the medium (amplitude-phase characteristics) in quasilinear and nonlinear formulations of this problem.

Development of the model of a medium layer motion with a nonlinear elastic stress component.

To build models in this case, it was used a mathematical apparatus of special Ateb-functions, which includes a certain variable, which to describe certain physical processes can be interpreted as an indicator of nonlinearity (it depends on the physical and mechanical properties of the working environment). Using Ateb functions, real physical vibrational processes can be described nonlinearly mathematically and therefore most adequately.

As an example, consider the model of the vibrating system environment in its contact with the working body of the vibrating system in the form of rigidly fixed beams. Using assumptions 3 and 4 of the proposed hypothesis of the motion of the medium, the equation with a nonlinear elastic stress component (6) is transformed into the following:

$$u_{tt} - \alpha^2 \left(u_{\xi\xi}\right)^{\nu+1} = \varepsilon G(u, u_t, \ldots u_{\xi\xi t}, \mu t), \quad (7)$$

where $G(u, u_t, \ldots u_{\xi\xi t}, \mu t) = \left[b_1 \sin \mu t - \vartheta u_t - \delta u_t u^2 + \beta u_{\xi\xi t}\right]$

Differential equation (7) is a mathematical model of the medium layer motion in the working container. At rigid contact of the environment and the container the boundary conditions are fair:

$$u(\xi, t)\big|_{\xi=0} = u(\xi, t)\big|_{\xi=l} = 0$$

The solution of equation (7) allows us to establish the law of motion of the medium layer – $u(\xi, t)$

Solving method

It is known that in real mechanical systems with many degrees of freedom, as well as in systems with distributed parameters, the presence of friction forces (both external and internal) leads to rapid damping of high-frequency oscillations and the establishment of a dynamic process with a single frequency. Therefore, in these systems it is advisable to consider the so-called single-frequency modes of oscillations with a frequency equal to the first (main frequency). The latter greatly facilitates the method of studying the perturbed equation (7). Let's build single-frequency solutions for it. It is easy to check that the undisturbed equation (it describes from a physical point of view the oscillations of the medium – in the absence of external forces on the vibration system, mathematically – the right part of equation (7) is zero), which corresponds to (7), ie equation

$$u_{tt} - \alpha^2 \left(u_{\xi\xi}\right)^{\nu+1} = 0, \quad (8)$$

allows the use of the Fourier variable separation method when finding its solution.

Representing according to the specified method function $u(\xi, t)$ as $u(\xi, t) = \Xi(\xi)T(t)$ for finding unknown functions $\Xi(\xi)$ and $T(t)$ nonlinear differential equations will be obtained:

$$\begin{aligned} &\frac{d^2\Xi}{d\xi^2}\left(\frac{d\Xi}{d\xi}\right)^{\nu} + \lambda\Xi(\xi) = 0, \\ &\frac{d^2T}{dt^2} + \alpha^2(\nu+1)\lambda T^{\nu+1}(t) = 0, \end{aligned} \quad (9)$$

where λ – невідомий параметр, який буде визначений нижче.

Based on (8), the function $\Xi(\xi)$ in (9) must meet the boundary conditions:

$$\Xi(0) = \Xi(l) = 0 \quad (10)$$

Linearly independent solutions of a differential equation for a function $\Xi(\xi)$ are determined using special periodic Ateb-functions [5] in the form of:

$$\Xi(\xi) = \Xi_0 \left\{ \begin{array}{l} sa\left(1, \dfrac{1}{\nu+1}, \left(\lambda \dfrac{\nu+2}{2\Xi_0}\nu\right)^{\frac{1}{\nu+2}} \xi\right) \\ ca\left(1, \dfrac{1}{\nu+1}, \left(\lambda \dfrac{\nu+2}{2\Xi_0}\nu\right)^{\frac{1}{\nu+2}} \xi\right) \end{array} \right., \quad (11)$$

where Ξ_0 is integration constant.

In (7) and below the parameter $\nu+1$ satisfies the condition $\nu+1=\frac{2n+1}{2m+1}$, where $m,n=0,1,2,.....$, that is $\nu>-1$ (at $\nu=0$ nonlinear model of oscillations of the medium layer becomes linear). The latter does not significantly narrow the set of parameter ν values, because, it can always be replaced with sufficient accuracy by the specified rational dependence.

Given (11) and boundary conditions (10), we'll find λ and the solution of the boundary value problem for the function $\Xi(\xi)$ in the form:

$$\lambda=\frac{2\Xi_0^{\nu}}{\nu+2}\left(\Pi_\xi\frac{1}{l}\right)^{\nu+2}, \tag{12}$$

$$\Xi(\xi)=\Xi_0 sa\left(1,\frac{1}{\nu+1},\Pi_\xi\frac{1}{l}\xi\right), \tag{13}$$

where $2\Pi_\xi$ is a period of the used special Ateb functions, ie $2\Pi_\xi=2\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\nu+1}{\nu+2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu+1}{\nu+2}\right)}$, $\Pi_\xi=\frac{\sqrt{\pi}\Gamma\left(\frac{\nu+1}{\nu+2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu+1}{\nu+2}\right)}$, $\Gamma(...)$ is the gamma function of the corresponding argument.

Similarly, the solution of the equation for the function $T(t)$ taking into account (12), takes the form:

$$T(t)=T_0 ca(\nu+1,1,\omega^*(a)t), \tag{14}$$

where $\omega^{*2}(a)=\alpha^2 a^{\nu}\left(\frac{\Pi_\xi}{l}\right)^{\nu+2}$, T_0 – constant, $a=\Xi_0 T_0$. Thus, from (12) and (13) we obtain the natural oscillations of the model layer in the case $G=0$, (single-frequency solution of the boundary value problem for the undisturbed equation (8)), which can be written as:

$$u(\xi,t)=a\cdot sa\left(1,\frac{1}{\nu+1},\Pi_\xi\frac{\xi}{l}\right)ca(\nu+1,1,\psi), \tag{15}$$

where $\psi=\omega^*(a)t+\theta$, and θ is constant.

We will also look for the solution of the perturbed equation (7) in the form (15), only for this case according to the averaging method [6] a and θ will already be functions of parameter t, that is $a=a(t)$ and $\theta=\theta(t)$. Given the latter, to determine $a(t)$ and $\theta(t)$ it will be obtained [7], for the non resonant case $\left(\omega(a)\neq\frac{\Pi_T}{\pi}\mu\right)$ system of differential equations of the environment model layer's motion:

$$\begin{aligned}\dot a&=\frac{\varepsilon}{\omega^*(a)p}\int_0^l\left(\begin{gathered}sa(1,\nu+1,\psi)\times\\ \times sa\left(1,\frac{1}{\nu+1},\frac{\Pi_\xi}{l}\xi\right)\times F_1(a,\xi,\psi,\gamma)\end{gathered}\right)dx,,\\ \dot\theta&=\frac{\varepsilon(\nu+2)}{2a\omega^*(a)p}\times\int_0^l\left(\begin{gathered}ca(\nu+1,1,\psi)\times\\ \times sa\left(1,\frac{1}{\nu+1},\frac{\Pi_\xi}{l}\xi\right)F_1(a,\xi,\psi,\gamma)\end{gathered}\right)dx\end{aligned} \tag{16}$$

where

$$P=\Xi_0{}^2\int_0^l\Xi_0{}^2(\xi)d\xi=\int_0^l sa^2\left(1,\frac{1}{\nu+1},\frac{\Pi_\xi}{l}\xi\right)d\xi=\frac{l\Gamma\left(\frac{1}{2}+\frac{\nu+1}{\nu+2}\right)}{2\Gamma\left(\frac{3}{2}+\frac{\nu+1}{\nu+2}\right)},$$

$$F_1(a,\xi,\psi,\gamma)=F(u,u_t,...\gamma)\Big|_{u=a\cdot ca(\nu+1,1,\psi)sa\left(1,\frac{1}{\nu+1},\frac{\Pi_\xi}{l}\xi\right)},\ \gamma=\mu t$$

Given that the right-hand sides of differential equation (16) are periodic functions of the parameters ψ and γ and the amplitude and phase of the oscillations θ for the period varies slightly, they can be written for the nonresonance case in the form:

$$\dot{a}=\frac{\varepsilon}{4\Pi_T P\omega^*(a)\pi}\int_0^l\int_0^{2\Pi_T}\int_0^{2\pi}\left(sa(1,\nu+1,\psi)\times sa\left(1,\frac{1}{\nu+1},\frac{\Pi_\xi}{l}\xi\right)F_1(a,\xi,\psi,\gamma)\right)d\gamma d\psi dx\,,$$
$$\dot{\theta}=\frac{\varepsilon(\nu+2)}{8\Pi_T Pa\omega^*(a)\pi}\int_0^l\int_0^{2\Pi_T}\int_0^{2\pi}\left(ca(\nu+1,1,\psi)\cdot sa\left(1,\frac{1}{\nu+1},\frac{\Pi_\xi}{l}\xi\right)F_1(a,\xi,\psi,\gamma)\right)d\gamma d\psi dx \tag{17}$$

Taking into account the accepted hypotheses of the motion of the medium, the system of differential equations (17) to determine its AFX (amplitude-phase characteristic) takes the form:

$$\dot{a}=\frac{2\varepsilon\Gamma\left(\frac{1}{2}+\frac{1}{\nu+2}\right)\Gamma\left(\frac{3}{2}+\frac{\nu+1}{\nu+2}\right)}{l\sqrt{\pi}\Gamma\left(\frac{1}{\nu+2}\right)\Gamma\left(\frac{1}{2}+\frac{\nu+1}{\nu+2}\right)}\left\{\frac{8a\pi\Gamma^2\left(1+\frac{\nu}{2}\right)\Gamma^2\left(\frac{1}{\nu+2}\right)}{(\nu+2)^2 l\Gamma^2\left(\frac{3}{2}+\frac{1}{\nu+2}\right)}-\frac{4\vartheta_1\sqrt{\pi}apl\Gamma\left(\frac{1}{\nu+2}\right)}{(\nu+2)\Gamma\left(\frac{3}{2}+\frac{1}{\nu+2}\right)}-\right.,$$
$$\left.-\frac{3\delta_1\pi a^3\Gamma\left(\frac{3}{\nu+2}\right)\Gamma\left(\frac{\nu+1}{\nu+2}\right)}{8\Gamma\left(\frac{3}{2}+\frac{3}{\nu+2}\right)\Gamma\left(\frac{5}{2}+\frac{\nu+1}{\nu+2}\right)}\times\frac{l}{\Pi_\xi}\right\},\quad \dot{\theta}=0$$

From the obtained relations it follows that in the first approximation for the amplitude of oscillations there are two stationary fixed values, which are respectively equal:

$$a_1=0\,,$$
$$a_2=\sqrt{\frac{\left\{\frac{8\pi\Gamma^2\left(1+\frac{\nu}{2}\right)\Gamma^2\left(\frac{1}{\nu+2}\right)}{(\nu+2)^2 l\Gamma^2\left(\frac{3}{2}+\frac{1}{\nu+2}\right)}-\frac{4\vartheta_1\sqrt{\pi}pl\Gamma\left(\frac{1}{\nu+2}\right)}{(\nu+2)\Gamma\left(\frac{3}{2}+\frac{1}{\nu+2}\right)}\right\}}{\left(\frac{3\delta_1\pi\Gamma\left(\frac{3}{\nu+2}\right)\Gamma\left(\frac{\nu+1}{\nu+2}\right)}{8\Gamma\left(\frac{3}{2}+\frac{3}{\nu+2}\right)\Gamma\left(\frac{5}{2}+\frac{\nu+1}{\nu+2}\right)}\cdot\frac{l}{\Pi_\xi}\right)}}\,.$$

The first stationary value corresponds to the relative calm of the environment, the second - a steady dynamic process, which is stable at $A'(a_2)<0$ and unstable in the opposite case.

Conclusions

Based on the Voigt's law the elastic-viscous model of the working environment of the vibrating system, which is presented as a continuous and homogeneous layer of flat elastic-plastic beams, the thickness of which is much less than their width. Solutions of the obtained equations of motion of the working medium layer for the stationary (steady) mode of its motion are constructed. In order to describe the physical and mechanical properties of different types of bulk media in the model, a power law (or close to it) of the relationship between the deformation of the medium layer and the stress in it is proposed. Accordingly, the nonlinearity index is introduced: a) into the elastic stress component in Voigt's law, b) into the viscous stress component in Voigt's law. Nonlinear equations for describing the oscillations of an arbitrary layer of the vibrating system medium are constructed. Using the asymptotic methods of nonlinear mechanics and special Ateb functions, the solutions of these equations are obtained.

The nonlinear model for describing the dynamics of the working environment of vibrating systems with boundary conditions of rigid contact of the environment with the vibrating system is more flexible, as the nonlinearity of the model, which depends on the type of working environment, significantly affects the oscillating process. increase the level of adequacy of the analytical model to the physics of the studied vibration process.

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